Principle of Induction

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The most basic fact about the set N of natural numbers is

Well Ordering Property. *Any nonempty subset A of $\mathbb{N}$ has a least element in A.*

An immediate corollary is the following version of the principle of mathematical induction.

Theorem 1 (Principle of Mathematical Induction)**.** *Let S be any subset of $\mathbb{N}$ such that (i) $1 \in S$ and (ii) if $k \in S$, then $k+1 \in S$. Then $S = \mathbb{N}$.*

Proof. Suppose $S \neq \mathbb{N}$. Then $A := \mathbb{N} \setminus S \neq \emptyset$. By the well-ordering principle, A has a least element, say, n. Note that $n > 1$, since $1 \in S$. Therefore, $n - 1 \in \mathbb{N}$. Clearly, $n - 1 \notin A$ and hence $n - 1 \in S$. By the hypothesis on S, $(n-1) + 1 = n \in S$, contradicting the fact that $n \in A$. This contradiction implies that $A = \emptyset$ or $S = \mathbb{N}$. □

Using this we derive the more usual version of the principle of induction.

Theorem 2 (Principle of Mathematical Induction)**.** *Let $P(n)$ be a statement involving the positive integer n. Assume that*

1. *$P(1)$ is true.*
2. *If $P(k)$ is true, then $P(k+1)$ is also true.*

Then $P(n)$ is true for all positive integers n. □

Proof. Let $S := \{n \in \mathbb{N} : \text{The statement } P(n) \text{ is true.}\}$. Then by hypothesis, $1 \in S$ and if $k \in S$, then $k+1 \in S$. Hence $S = \mathbb{N}$. That is, $P(n)$ is true for all $n \in \mathbb{N}$. □

Remark 3. The condition 1) of Theorem 2 is called the basis for induction and 2) is called the inductive step.

Both these conditions are important can seen from Ex. 11 and Ex. 12.

We illustrate the principle by means of a few examples.

Example 4. For any positive integer n, we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

Let $P(n)$ be the statement Eq. 1. We first observe that $P(1)$ is true. Let us now assume that $P(k)$ is true. Thus we have $1 + \cdots + n = n(n+1)/2$. We now add $n+1$ to both sides. We then get

$$1 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

The right side of this equation is

$$\frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2},$$

which is $P(k+1)$. Hence $P(k+1)$ is true. Thus, by principle of induction, we see that Eq. 1 is true for all $n \in \mathbb{N}$. ☐

Ex. 5. For any positive integer n we have

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Ex. 6. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$, for all $n \in \mathbb{N}$.

Example 7 (Bernoulli's Inequality)**.** Let $x \in \mathbb{R}$ such that $x > -1$. Then

$$(1+x)^n \geq 1 + nx, x > -1 \text{ and } n \in \mathbb{N}. \tag{2}$$

Let $P(n)$ be the statement Eq. 2.

If $k = 1$, $P(1)$ is clearly true. Assume that $P(k)$ is true. That is, we have $(1+x)^k \geq 1+kx$. Since $1 + x > 0$ we multiply both sides of the inequality by $1 + x$ and get

$$\begin{aligned} (1+x)^{k+1} = (1+x)(1+x)^k &\geq (1+x)(1+kx) \\ &\geq 1 + (k+1) + kx^2 \\ &\geq 1 + (k+1)x. \end{aligned}$$

This shows that $P(k+1)$ holds. Hence, by the principle of induction, Eq. 2 holds true for all $n \in \mathbb{N}$. ☐

Example 8. If $a_1, a_2, \ldots, a_n$ are positive integers relatively prime to another integer b, then their product $a_1 \cdots a_n$ is relatively prime to b.

Recall that we say two integers r and s are said to be relatively prime if the only positive integral divisor of both is 1. Equivalently, r and s are relatively prime iff there exist integers a and b such that $ar + bs = 1$.

What is $P(n)$ here? We let $P(n)$ be the statement that if we are given n integers whcih are relatively primt to b, so is their product.

$P(1)$ is certainly true by hypothesis. We now prove that $P(2)$ is also true. Let a_1 and a_2 be relatively prime to b. Then there exist integers x_j, y_j, $j = 1, 2$ such that $a_1x_1 + by_1 = 1$ and $a_2x_2 + by_2 = 1$. Multiplying these equations, we get

$$a_1a_2(x_1x_2) + b(a_1x_1y_2 + a_2x_2y_1) = 1.$$

This proves that a_1a_2 is relatively prime to b.

Assume that $P(k)$ is true. Let $a_1, \ldots, a_k, a_{k+1}$ be relatively prime to b. Since $P(k)$ is true, $a_1 \cdots a_k$ is relatively prime to b. Also, by assumption, a_{k+1} is relatively prime to b. By $P(2)$, the 0product $(a_1 \cdots a_k)a_{k+1}$ is relatively prime to b. That is, $P(k+1)$ holds. By induction principle, $P(n)$ holds true for all $n \in \mathbb{N}$. □

Remark 9. The crucial step in the above proof is to show that $P(2)$ holds. To appreciate this, do the next exercise.

Ex. 10. Find the fallacious step in the following argument. We prove by induction that any n things are the same. If $n = 1$ this is clear. Assume that any k things are the same. Let $x_1, x_2, \ldots, x_k, x_{k+1}$ be given. By induction hypothesis (that is another way of saying $P(k)$ is true) applied the the k things $a_1, \ldots, a_k$ we find that $a_1 = a_2 = \cdots = a_k$. Similarly, we conclude that $a_2 = a_3 = \cdots = a_k = a_{k+1}$. Hence $a_1 = a_2 = \cdots = a_k = a_{k+1}$. Hence by induction principle, given any n things, they are always the same.

Ex. 11. Show that the statement $2 + 2 + \cdots + 2n = (n+2)(n-1)$ for all $n \in \mathbb{N}$ satisfies the inductive step but has no basis.

Ex. 12. Show that for some values of n, $n^2 + n + 41$ is a prime number for some it is not, so that there is no inductive step which would show that $n^2 + n + 41$ is a prime number for all n.

Ex. 13. Derive the formula for the sum of first n terms of an arithmetic progression:

$$a + (a+d) + (a+2d) + \cdots + (a + (n-1)d) = \frac{n[2a + (n-1)d]}{2}.$$

Ex. 14. Derive the formula for the sum of first n terms of a geometric progression:

$$a + (ar) + (ar^2) + \cdots + (ar^{n-1}) = \frac{a(r^{n-1})}{r-1}, \qquad r \neq 1.$$

We now give a set of exercises for practice. In all these problems the statement is to be proved for all positive integers by induction.

Ex. 15. If $a \equiv b|m|$, then

$$a^n \equiv b^n|m|.$$

Ex. 16. $4^n \equiv 3n + 1|9|$.

Ex. 17. $7^n \equiv 6n + 1|36|$.

Ex. 18. $2^n + 3^n \equiv 5^n|6|$.

Ex. 19. $2 \cdot 7^n \equiv 2^n(2 + 5n)|25|$.

Ex. 20. 6 divides $n(n^2 + 3n + 8)$ and $n^3 - n$.

Ex. 21. $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$.

Ex. 22. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

Ex. 23. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$.

Ex. 24. $(-4)^n \equiv 1 - 5n|25|$.

Ex. 25. $2^{2^n} + 1 \equiv 5|12|$.

Ex. 26. $(\frac{1}{2} + 1)(\frac{1}{2} + \frac{1}{2})(\frac{1}{2} + \frac{1}{3}) \cdots (\frac{1}{2} + \frac{1}{n}) = \frac{(n+1)(n+2)}{2^{n+1}}$.

Ex. 27. $2^{2n-1}(n!)^2 \geq (2n)!$.

Ex. 28. $(2n)! \geq (n!)^2 2^n$.

Ex. 29. $8^n \mid (4n)!$.

Ex. 30. $2^{n+2} \mid (2n+3)!$.

Ex. 31. $31 \mid 2^{5n-1}$.

Ex. 32. $17 \mid 2^{8n-4} + 1$.

Ex. 33. Fibonacci numbers. We define a sequence of numbers as follows: $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-2} + f_{n-1}$ for all $n \geq 3$. The number f_n is called the nth Fibonacci number.

Prove that, for each $n \in \mathbb{N}$, the following hold:

$$\begin{aligned} f_2 + f_4 + \cdots + f_{2n} &= f_{2n+1} - 1. \\ f_1 + f_3 + \cdots + f_{2n-1} &= f_{2n}. \\ f_1 + f_2 + \cdots + f_n &= f_{n+2} - 1 \\ f_n &< 2^n. \end{aligned}$$

Ex. 34. If A is the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$ for all $n \in \mathbb{N}$.

For certain applications, we need a (seemingly) stronger version of the induction principle.