Lebesgue Integral without Measure Theory

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

This article gives a very brief account of the theory of Lebesgue integral which avoids a lengthy treatment of measure theory. The omitted proofs are either standard or easy. This article Follows closely Van Daele and Riesz-Nagy' treatise on Functional Analysis. Even when one write all the details, the article may swell up to 8 pages, in any case, it will be well within 10 pages!

One quickly reviews the following three items:

1. Integral $I(s)$ of step functions.
2. Extension of the integral I to $C[a,b]$ and the standard properties.
3. Integral I on $K(\mathbb{R})$, the space of continuous functions with compact support.

Definition 1. Let $f\colon \mathbb{R} \to \mathbb{C}$ be function. Define $\| f \|$, taking values in $[0, \infty]$, by setting

$$\| f \| := \inf \left\{ \sum_{n=1}^{\infty} I(g_n) : g_n \in K(\mathbb{R}),\ g_n \geq 0 \text{ and } |f(t)| \leq \sum_n g_n(t),\ \text{for all } t \in \mathbb{R} \right\}.$$

The properties of this map $\| \ \| : K(\mathbb{R}) \to [0, \infty]$ are given in the next porposition.

Proposition 2. *Let $f_n\colon \mathbb{R} \to \mathbb{C}$, $n \in \mathbb{N}$, be functions and $\lambda \in \mathbb{C}$. Then*
(i) *We have $\| f \| = I(|f|)$ for $f \in K(\mathbb{R})$.*
(ii) $\| \lambda f \| = |\lambda| \, \| f \|$.
(iii) *If $|f(t)| \leq \sum_n f_n(t)$ for $t \in \mathbb{R}$, then $\| f \| \leq \sum_n \| f_n \|$.*

Proof. Idea of the proof: Only the first one requires proof, which is an easy application of Dini's theorem (on uniform convergence).

It is clear that $\| f \| \leq I(|f|)$. Let (g_n) be a sequence as in the definition of $\| f \|$. Define h_n as follows:

$$h_0 = 0;\ h_n = \min \left(|f|, \sum_{k=1}^{n} g_k \right).$$

The functions h_n are positive continuous functions with compact support as their supports are contained in the support of f. The sequence (h_n) increases to $|f|$. We can apply Dini's theorem to conclude that h_n converges uniformly to $|f|$. Hence $I(h_n) \to I(|f|)$ (by an elementary result in Riemann integration). If we let $\varphi_n := h_n - h_{n-1}$, then the sequence (φ_n) has the properties stated for functions g_n that appear in the definition of $\| f \|$. For all x, we have $\varphi_n(x) \le h_n(x)$ and therefore $\sum_n I(\varphi_n) \le \sum_n I(g_n)$.

The rest of the properties are easily seen. □

Definition 3. Let $\mathcal{L}^1(\mathbb{R})$ denote the set of functions $f\colon \mathbb{R} \to \mathbb{C}$ such that for a given $\varepsilon > 0$, there exists $g \in K(\mathbb{R})$ such that $\| f - g \| < \varepsilon$.

If $f \in \mathcal{L}^1(\mathbb{R})$, we define $I(f) := \lim I(f_n)$, where $f_n \in K(\mathbb{R})$ is such that $\| f - f_n \| \to 0$.

Proposition 4. *The following hold:*
(a) $\mathcal{L}^1(\mathbb{R})$ *is a complex vector space.*
(b) *If* $f \in \mathcal{L}^1(\mathbb{R})$*, so do* $\overline{f}, |f|$.
(c) $I(f)$ *is well-defined.*
(d) I *extends the Riemann integral on* $K(\mathbb{R})$ *to* $\mathcal{L}^1(\mathbb{R})$.
(e) I *is linear and positive (that is,* $I(f) \ge 0$ *if* $f \ge 0$*) on* $\mathcal{L}^1(\mathbb{R})$.
(f) $I(|f|) = \| f \|$ *for* $f \in \mathcal{L}^1(\mathbb{R})$. □

Theorem 5 (Completeness of $\mathcal{L}^1(\mathbb{R})$)**.** *If* (f_n) *is a sequence in* $\mathcal{L}^1(\mathbb{R})$ *such that* $\| f_n - f_m \| \to 0$ *as* $n, m \to \infty$*, then there exists an* $f \in \mathcal{L}^1(\mathbb{R})$ *such that* $\| f - f_n \| \to 0$.

Proof. Let (f_n) be as in the statement. We may assume, going to a subsequence if necessary, that $\| f_{n+1} - f_n \| < 2^{-n}$ for all n. Define

$$f(t) := \begin{cases} \lim f_n(t), & \text{if it exists} \\ 0, & \text{otherwise.} \end{cases}$$

Fix $n \in N$. Let $t \in \mathbb{R}$ be such that $\lim f_n(t)$ exists. Then we have

$$f(t) - f_n(t) = \sum_{k=n}^{\infty} (f_{k+1}(t) - f_k(t)), \tag{1}$$

so that

$$|f(t) - f_n(t)| \le \sum_{k=n}^{\infty} |f_{k+1}(t) - f_k(t)|. \tag{2}$$

If the last (absolute) sum is finite, the first sum is finite and hence $\lim_n f_n(t)$ exists. On the other hand, if $\lim_n f_n(t)$ does not exist, the absolute sum is infinite and hence the inequality (2) is trivially true. By (iii) of Poroposition 2, we get

$$\| f - f_n \| \le \sum_{k=n}^{\infty} \| f_{k+1} - f_k \| \le 2^{1-n}.$$

This implies that $f \in \mathcal{L}^1(\mathbb{R})$ (why?) and that $f_n \to f$ in $\mathcal{L}^1(\mathbb{R})$. □

Theorem 6 (Monotone Convergence Theorem)**.** *Let* (f_n) *be as sequence in* $\mathcal{L}^1(\mathbb{R})$ *such that (i)* $f_n \ge 0$*, (ii)* $f_n \le f_{n+1}$ *for* $n \in \mathbb{N}$ *and (iii)* $\sup I(f_n) < \infty$*. If* $f_n \to f$ *pointwise, then* $f \in \mathcal{L}^1(\mathbb{R})$ *and we have* $I(f) = \lim I(f_n)$.

Proof. As in the proof of the last theorem, we have

$$\begin{aligned} \| f - f_n \| &\leq \sum_{k=n}^{\infty} \| f_{k+1} - f_k \| \\ &= \sum_{k=n}^{\infty} I(f_{k+1} - f_k) \\ &= \lim_m I(f_m) - I(f_n). \end{aligned} \tag{3}$$

The RHS of (3) converges to 0 as $n \to \infty$. It follows that $f \in \mathcal{L}^1(\mathbb{R})$. We also have

$$\| f - f_n \| = I(f - f_n) = I(f) - I(f_n),$$

so that we obtain $I(f) = \lim_n I(f_n)$. □

Theorem 7 (Dominated Convergence Theorem)**.** *Let (f_n) be a sequence in $\mathcal{L}^1(\mathbb{R})$ such that $\lim f_n(t)$ exists for all $t \in \mathbb{R}$. Let f be the limit function. Assume that there exists $g \in \mathcal{L}^1(\mathbb{R})$ such that $|f_n(t)| \leq g(t)$ for $t \in \mathbb{R}$. Then $f \in \mathcal{L}^1(\mathbb{R})$ and we have $I(f) = \lim_n I(f_n)$.*

Proof. This is deduced from MCT by the usual method. See my notes on Lebesgue Integral—Daniell's approach for details. □

Definition 8. We say that a function $f \colon \mathbb{R} \to \mathbb{C}$ is *integrable* if $f \in \mathcal{L}^1(\mathbb{R})$. Note that for any integrable function $\| f \| < \infty$.

Example 9. For any (finite) interval $J := (a, b), [a, b], (a, b], [a, b)$, the characteristic function $\mathbf{1}_J$ is integrable. Any step function on a compact interval is integrable.

Definition 10. A subset $E \subset \mathbb{R}$ is said to be a *null set* or a set of measure zero if the characteristic function $\mathbf{1}_E \in \mathcal{L}^1(\mathbb{R})$ and $I(\mathbf{1}_E) = 0$.

Proposition 11. *The following are true:*

(a) *A subset of a null set is a null set.*
(b) *A countable union of null sets is a null set.*
(c) *A singleton subset is negligible and hence any countable subset of $\mathbb{R}$ is a null set.*
(d) *If $I(f) = 0$, then the set $\{x : f(x) \neq 0\}$ is a null set.* □

Definition 12. (i) We say that a property P is true almost everywhere if the set of points where P is not true is a null set.

(ii) We say that a subset $E \subset \mathbb{R}$ is integrable iff $\mathbf{1}_E \in \mathcal{L}^1(\mathbb{R})$.

(iii) We say that subset $E \subset \mathbb{R}$ is *measurable* iff $E \cap K$ is integrable for all compact subsets of $\mathbb{R}$.

Proposition 13. *The following hold:*

(a) *The union of a countable family of measurable subsets is measurable.*
(b) *The intersection of a countable family of measurable subsets is measurable.*
(c) *The complement of a measurable set is measurable.*
(d) *Any interval is measurable.*
(e) *All open sets and all closed sets are measurable.* □

Ex. 14. Define $\| f \|_2 := \left\| f^2 \right\|^{1/2}$ for $f\colon \mathbb{R} \to \mathbb{C}$. Prove the following: (i) If $\| f \|_2 < \infty$ and $\| g \|_2 < \infty$, then $\| fg \| < \| f \|_2 \, \| g \|_2$, (ii) $\| \lambda f \|_2 \leq |\lambda| \, \| f \|_2$, and (iii) if $|f(t)| \leq \sum |f_n|$, then $\| f \|_2 \leq \sum_n \| f_n \|_2$. Define $\mathcal{L}^2(\mathbb{R})$ and show that it is complete with respect to the seminorm $\| \ \|_2$.