Isometries of $\mathbb{R}^n$

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Let (X, d) and (y, d) be metric spaces. A map $f\colon X \to Y$ is said to be an isometry if $d(f(x_1), f(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$. Note that an isometry is always one-one but in general not onto. For instance, consider $f\colon [1, \infty) \to [1, \infty)$ given by $f(x) = x + 1$. If f and g are isometries of X to itself, then $g \circ f$ is also an isometry. The set of surjective isometries of a metric space form a group under the composition.

We consider $\mathbb{R}^n$ with the Euclidean inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. We then have the notion of norm or length of a vector $\| x\| := \sqrt{\langle x, x\rangle}$. It is well-known that $d(x, y) := \| x - y\|$ defines a metric on X. Thus $(\mathbb{R}^n, d)$ becomes a metric space. The aim of this article is to give a complete description of all isometries of $\mathbb{R}^n$ and look a little more geometrically into the isometries of $\mathbb{R}^3$.

First a bit of convention: We consider $\mathbb{R}^n$ as the vector space of column vectors, that is, $n \times 1$ real matrices. Given an $n \times n$ matrix A, we have a linear map on $\mathbb{R}^n$ given by $x \mapsto Ax$. In the sequel, we shall not distinguish between the matrix A and the associated linear map. I am sure that the context will make it clear what we are referring to.

Let us first of all look at some examples of isometries. For a fixed $v \in \mathbb{R}^n$, consider the translation $T_v\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T_v(x) := x + v$. Then T_v is an isometry of $\mathbb{R}^n$:

$$d(T_v x, T_v y) = \| T_v x - T_v y\| = \| (x + v) - (y + v)\| = \| x - y\| = d(x, y).$$

We now describe another important class of isometries of $\mathbb{R}^n$. A linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be orthogonal if it preserves inner products: for every pair $x, y \in \mathbb{R}^n$, we have

$$\langle f(x), f(y)\rangle = \langle x, y\rangle .$$

The following result is well-known. For a proof, I refer the reader to my book on linear algebra.

Theorem 1. *For a linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ the following are equivalent:*
1. *f is orthogonal.*
2. *$\| f(x)\| = \| x\|$ for all $x \in \mathbb{R}^n$.*
3. *f takes an orthonormal basis of $\mathbb{R}^n$ to an orthonormal basis.* □

We now look at a very special class of orthogonal maps.

Fix a unit vector $u \in \mathbb{R}^n$. Let $W := (\mathbb{R}u)^\perp := \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$. Then W is a vector subspace of dimension $n-1$. This can be seen as follows. The map $f_u \colon \mathbb{R}^n \to \mathbb{R}$ given by $f_u(x) := \langle x, u \rangle$ is linear. It is nonzero, since $f_u(u) = 1$. Hence the image of f_u is a nonzero vector subspace and hence all of $\mathbb{R}$. Also, we observe that $W = \ker f_u$. Hence by the rank-nullity theorem,

$$n = \dim \mathbb{R}^n = \dim \ker f_u + \dim \operatorname{Im} f_u.$$

It follows that W is an $n-1$ dimensional vector subspace of $\mathbb{R}^n$. We thus have an orthogonal decomposition $\mathbb{R}^n = W \oplus \mathbb{R}u$. We use W as a mirror and reflect across it. Thus any vector in W is mapped to itself whereas the vector u is mapped to $-u$. Thus the reflection $\rho_W \colon \mathbb{R}^n \to \mathbb{R}^n$ is given by $\rho(x) = w - tu$, where $x = w + tu$, $t \in \mathbb{R}$. This map is clearly an orthogonal linear map. For, if $\{w_1, \ldots, w_{n-1}\}$ is an orthonormal basis of W, then $\{w_1, \ldots, w_{n-1}, u\}$ is an orthonormal basis of $\mathbb{R}^n$. The linear map ρ_W carries this orthonormal basis to the orthonormal basis $\{w_1, \ldots, w_{n-1}, -u\}$ and hence is orthogonal. We have another description of this map as follows:

$$\rho_W(x) := x - 2\langle x, u \rangle u. \tag{1}$$

The advantage of this expression is that it is basis-free. We also observe that the expression remains the same if we replace u by $-u$.

We shall look at the case when $n = 2$ in detail. We shall derive a matrix representation of ρ_W from (1). Consider the one dimensional subspace $\mathbb{R}\begin{pmatrix}\cos t \\ \sin t\end{pmatrix}$. Let $u = \begin{pmatrix}-\sin t \\ \cos t\end{pmatrix}$. Then

$$\begin{aligned}
\rho_W(e_1) &= \begin{pmatrix}1 \\ 0\end{pmatrix} - 2\left\langle \begin{pmatrix}1 \\ 0\end{pmatrix}, \begin{pmatrix}-\sin t \\ \cos t\end{pmatrix} \right\rangle \begin{pmatrix}-\sin t \\ \cos t\end{pmatrix} \\
&= \begin{pmatrix}1 - 2\sin^2 t \\ 2\sin t\cos t\end{pmatrix} = \begin{pmatrix}\cos 2t \\ \sin 2t\end{pmatrix} \\
\rho_W(e_2) &= \begin{pmatrix}0 \\ 1\end{pmatrix} - 2\left\langle \begin{pmatrix}0 \\ 1\end{pmatrix}, \begin{pmatrix}-\sin t \\ \cos t\end{pmatrix} \right\rangle \begin{pmatrix}-\sin t \\ \cos t\end{pmatrix} \\
&= \begin{pmatrix}\sin 2t \\ -\cos 2t\end{pmatrix}.
\end{aligned}$$

Thus the reflection about the line is given by $\rho_W = \begin{pmatrix}\cos 2t & \sin 2t \\ \sin 2t & -\cos 2t\end{pmatrix}$.

Remark 2. Reflections generate the group of orthogonal linear maps. (We shall not prove this.) Thus, they are the building blocks of orthogonal linear maps. If we observe that $\rho^2 = 1$ for any reflection, then the analogy between the transpositions and reflections is striking.

A surprising result is

Theorem 3. *Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be an isometry with $f(0) = 0$. Then f is an orthogonal linear map.* □

A proof of this can also be found in my book. However, we indicate a slightly different proof found in an unpublished note by Artin.

Lemma 4. *Let $x, y \in \mathbb{R}$. Assume that $\langle x, x \rangle = \langle x, y \rangle = \langle y, y \rangle$. Then $x = y$.*

Proof. We compute the length square of $x - y$:

$$\langle x-y, x-y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle = 0,$$

in view of our hypothesis. Hence $x = y$. □

Lemma 5. *An isometry of $\mathbb{R}^n$ that fixes the origin preserves the inner products.*

Proof. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an isometry such that $f(0) = 0$. We need to prove that $\langle f(x), f(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$. Since $d(f(x), f(y)) = d(x, y)$, we have $\| f(x) - f(y)\| = \|x - y\|$. Taking squares, we get

$$\langle f(x) - f(y), f(x) - f(y)\rangle = \langle x - y, x - y\rangle\,, \text{for all } x, y \in \mathbb{R}^n. \tag{2}$$

Since $f(0) = 0$, setting $y = 0$ in (2), we get $\langle f(x), f(x)\rangle = \langle x, x\rangle$. Similarly, $\langle f(y), f(y)\rangle = \langle y, y\rangle$. Now, if we expand both sides of (2) and cancelling equal terms such as $\langle x, x\rangle$ and $\langle y, y\rangle$, we get the desired result. □

We now prove Theorem 3.

Proof of Theorem 3. We need only show that f is linear. Let $x, y \in \mathbb{R}^n$. Let $z = x + y$. We shall show first that $f(z) = f(x) + f(y)$. In view of Lemma 4, it suffices to show that

$$\langle f(z), f(z)\rangle = \langle f(z), f(x) + f(y)\rangle = \langle f(x) + f(y), f(x) + f(y)\rangle\,.$$

Expanding the last two terms, we need to show that

$$\langle f(z), f(z)\rangle = \langle f(z), f(x)\rangle + \langle f(z), f(y)\rangle = \langle f(x), f(x)\rangle + 2\langle f(x), f(y)\rangle + \langle f(y), f(y)\rangle\,. \tag{3}$$

Since f preserves inner products, (3) holds same iff

$$\langle z, z\rangle = \langle z, x\rangle + \langle z, y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle\,, \tag{4}$$

holds. Since $z = x + y$, (4) is clearly true!

Similarly, if we take $y = ax$, $a \in \mathbb{R}$, then we need to show that $f(y) = af(x)$. In view of Lemma 4, it is enough to show that

$$\langle f(ax), f(ax)\rangle = \langle f(ax), af(x)\rangle = \langle af(x), af(x)\rangle\,.$$

Since f is inner product preserving, it suffices to show that

$$\langle ax, ax\rangle = a\langle ax, x\rangle = a^2\langle x, x\rangle\,,$$

which is true. □

As immediate consequences of Theorem 3, we have

Corollary 6. *(i) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Then $f = T_v \circ A$ where $v = f(0)$.*
(ii) Any isometry of $\mathbb{R}^n$ is onto.

Proof. Consider $g := T_{-v} \circ f$ where $v = f(0)$. Then clearly, g is an isometry of $\mathbb{R}^n$ such that $g(0) = 0$. Hence g is an orthogonal linear map, say, A. Hence $f = T_v \circ A$. Thus (i) is proved.

Since any translation and any orthogonal linear map are onto, so is an isometry. □

We now give a complete list of all orthogonal maps of $\mathbb{R}^2$. Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal. If we represent A with respect to the standard orthonormal basis $\{e_1, e_2\}$ as a matrix, then A is either of the form

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \text{ or of the form } \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}.$$

In the first case, we say that it is a rotation by an angle t in the anticlockwise direction. In the latter case, it is reflection with respect to the line $\mathbb{R}\begin{pmatrix} \cos(t/2) \\ \sin(t/2) \end{pmatrix}$, as seen earlier.

Thus we have understood all the orthogonal maps of $\mathbb{R}^2$. We now wish to do the same in $\mathbb{R}^3$.

Definition 7. Given a two dimensional vector subspace W of $\mathbb{R}^3$, we define a rotation in the plane W as follows. Fix an orthonormal basis $\{w_1, w_2\}$ of $\mathbb{R}^3$. Fix a unit vector u such that $u \perp W$. We consider the linear map given by

$$\begin{aligned} R_{W,t}(w_1) &= \cos t\, w_1 + \sin t\, w_2 \\ R_{w,t}(w_2) &= -\sin t\, w_1 + \cos t\, w_2 \\ R_{w,t}(u) &= u. \end{aligned}$$

Clearly, $R_{W,t}$ is orthogonal. Also, with respect to the orthonormal basis $\{w_1, w_2, u\}$, it is represented by

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We shall refer to $R_{W,t}$ as the rotation in the plane W with $\mathbb{R}u$ as the axis of rotation. Note that $\det(R_{W,t}) = 1$.

Proposition 8. (i) *Let A be a 3×3 orthogonal matrix. Assume that* $\det A = 1$. *Then* 1 *is an eigenvalue of A.*
(ii) *Let A be a 3×3 orthogonal matrix. Assume that* $\det A = 1$. *Then A is a rotation.*

Proof. (i). Consider the following chain of equations:

$$\begin{aligned} \det(A - I) &= \det(A - AA^t) = \det(A(I - A^t)) \\ &= \det(A)\det(I - A^t) = \det((I - A^t)^t) = \det(I - A). \end{aligned}$$

For any $n \times n$ matrix B, we have $\det(-B) = (-1)^n \det B$. Hence, we see that $\det(A - I) = \det(I - A) = -\det(A - I)$. We conclude that $\det(A - I) = 0$. From this, (i) follows.

We now prove (ii). By (i), there exists a unit vector $u \in \mathbb{R}^3$ such that $Au = u$. Let $W := (\mathbb{R}u)^\perp$. Then W is a two dimensional vector subspace. We claim that $Aw \in W$ for any $w \in W$. This is seen as follows:

$$\langle Aw, u \rangle = \langle Aw, Au \rangle = \langle w, u \rangle = 0.$$

Thus $A|_W : W \to W$ is an orthogonal linear map. If we choose an orthonormal basis $\{w_1, w_2\}$ of W, then A can be represented with respect to the orthonormal basis $\{w_1, w_2, u\}$ either as

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or as } \begin{pmatrix} \cos t & \sin t & 0 \\ \sin t & -\cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The second case cannot occur as the determinant is -1, contradicting our hypothesis. Hence (ii) follows. □

Theorem 9. *Let A be any 3×3 orthogonal matrix. Then either A is a rotation or is a rotation followed by a reflection.*

Proof. If $\det A = 1$, then we know that it is a rotation. So we assume that $\det A = -1$. Then consider the reflection matrix corresponding to the xy plane: $\rho = \operatorname{diag}(1\ 1\ -1)$. Then $B = \rho A$ has determinant 1 and hence is a rotation. Since $A = \rho B$, the theorem follows. □

Remark 10. Let A be an orthogonal 3×3 matrix with $\det A = 1$. How can we decide whether A is a pure reflection or is a rotation followed by a reflection? If it is a pure reflection, then its eigenvalues are $+1, +1, -1$. In the other case, either all eigenvalues are -1 or it has only one real eigenvalue, namely -1.

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Reference

S. Kumaresan, *Linear Algebra—A Geometric Approach*, Prentice-Hall of India, 2000.