Functions of Bounded Variation

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The starting point for the study of functions of bounded variation is the desire to define the length of a curve. Let $\gamma\colon [a,b] \to \mathbb{R}^N$ be a continuous function. We call such a function as a curve in $\mathbb{R}^N$. If we write $x(t) := \gamma(t) = (x_1(t), \dots, x_N(t))$, then $x(t)$ may be considered as the position vector of a particle at the instant t. We wish to "compute" the length of the curve γ. Since we know how to compute the length of a line segment, we approximate the length of γ by the length of a polygonal path which approximates γ.

Let $P := \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a,b]$. Then we have a polygonal path which is built out of line segments joining $x(t_{j-1})$ to $x(t_j)$. Its length is $\sum_{j=1}^n \|x(t_j) - x(t_{j-1})\|$. We shall denote this length by $L(\gamma, P)$. Our intuition says that if we keep refining the partition, we shall get better approximation to the length of γ. This suggests that we define

$$\text{Length of } \gamma \equiv L(\gamma) := \operatorname{lub}\{L(\gamma, P) : P \text{ is a partition of } [a,b]\}.$$

It may happen that $L(\gamma)$ does not exist in $\mathbb{R}$. If $L(\gamma)$ exists, we say that the curve γ is rectifiable and call $L(\gamma)$ as its length.

Let us keep the notation as above. Fix $1 \le i \le N$. Note that $|x_i(t)| \le \|x(t)\|$. Hence it follows that $L(x_i, P) := \sum_j |x_i(t_j) - x_i(t_{j-1})| \le \sum_j \|x(t_j) - x(t_{j-1})\|$. Hence, if we set $L(x_i) := \operatorname{lub}\{L(x_i, P) : P \text{ is a partition of } [a,b]\}$, then $L(x_i) \le L(\gamma)$. Thus if γ is rectifiable, then $L(x_i)$ is finite for each i.

Conversely, if each $L(x_i)$ is finite, then γ is rectifiable. We defer the proof of this statement towards the end of this article. We also show (at the end) that if γ is continuous differentiable, then γ is rectifiable and that $L(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$.

Thus it behooves us to study the special case of real valued functions f on $[a,b]$ and discuss the so-called length $L(f)$.

Definition 1. Let $f\colon [a,b] \to \mathbb{R}$ be given. Let $P := \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a,b]$. The variation of f over $[a,b]$ with respect to the partition P is defined by

$$V(f, [a,b], P) := \sum_{j=1}^n |f(x_j) - f(x_{j-1})|.$$

If the interval $[a,b]$ is understood, we simplify the notation as $V(f, P)$.

We insert a point $x \in (x_{j-1}, x_j)$ to form a new partition $Q = \{x_0, \ldots, x_{j-1}, x, x_j, \ldots, x_n\}$. In the sum for $V(f, P)$ we replace the term $|f(x_j - f(x_{j-1})|$ by $|f(x_j - f(x)| + |f(x_j - f(x)|$. This shows that $V(f, P) \leq V(f, Q)$.

This leads us to conclude that $V(f, P) \leq V(f, Q)$ if Q is a refinement of P.

Definition 2. The variation f on $[a, b]$ is defined by

$$V(f, [a, b]) := \text{lub}\, \{V(f, [a, b], P) : \ P \text{ is a partition of } [a, b]\}.$$

We say that f is of bounded variation on $[a, b]$ if $V(f, [a, b])$ is finite.

Remark 3. If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$. Let $x \in [a, b]$ and consider the partition $P := \{a, x, b\}$. Then we have

$$|f(a) - f(x)| \leq |f(x) - f(a)| + |f(b) - f(x)| = V(f, P) \leq V(f, [a, b]).$$

By triangle inequality we obtain

$$|f(x)| = |f(x) - f(a) + f(a)| \leq |f(x) - f(a)| + |f(a)| \leq V(f, [a, b]) + |f(a)|\,.$$

Example 4. Any monotone function $f\colon [a, b] \to \mathbb{R}$ is of bounded variation and we have $V_a^b(f) = |f(b) - f(a)|$. Assume that f is increasing. Observe that $|f(x_j) - f(x_{j-1})| = f(x_j) - f(x_{j-1})$. Hence the sum $V(f, [a, b], P)$ is telescopic sum.

$$\sum_j |f(x_j) - f(x_{j-1})| = \sum_j f(x_j) - f(x_{j-1}) = f(b) - f(a).$$

Example 5. Let $f\colon [a, b] \to \mathbb{R}$ be Lipschitz, say, $|f(x) - f(y)| \leq L\,|x - y|$. Then f is of bounded variation on $[a, b]$. Observe that, for any partition, we have

$$\sum_j |f(x_j) - f(x_{j-1})| \leq \sum_j L(x_j - x_{j-1}) = L(b - a).$$

Example 6. Let $f\colon [a, b] \to \mathbb{R}$ be continuously differentiable (that is, $f'(x)$ exists on $[a, b]$ and f' is continuous on $[a, b]$). Then f is of bounded variation.

Since f' is continuous on $[a, b]$, there exists $M > 0$ such that $|f'(x)| \leq M$. For, $x, y \in [a, b]$, mean value theorem tells us that there exists z between x and y such that $f(x) - f(y) = f'(z)(x - y)$. It follows that $|f(x) - f(y)| \leq M\,|x - y|$. Thus, f is Lipschitz. The result follows from the last example.

Example 7. We now give an example of a bounded function which is not of bounded variation. Let $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ for $x \in [0, 1]$. Let $x_0 = 0$ and $x_j = \frac{2}{(n-j)\pi}$ for $0 < j < n$. We then have

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| = 2n.$$

Hence F is not of bounded variation.

Example 8. The function $f(x) = x\sin(1/x)$ for $0 < x \leq 2/\pi$ and $f(0) = 0$ is continuous, bounded but is not of bounded variation. (Draw the graph of this function to see why.)

Let

$$x_0 = 0 < x_1 = \frac{2}{(2n+1)\pi}, < x_2 = \frac{2}{(2n)\pi} < x_3 = \frac{2}{(2n1)\pi} \cdots < x_{2n} = \frac{2}{2\pi} < x_{2n+1} = \frac{2}{\pi}$$

be a partition of the domain. Then we have

$$\begin{aligned}\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| &= \frac{2}{\pi} + \frac{2}{2\pi} + \frac{2}{3\pi} + \cdots + \frac{2}{(2n+1)\pi} \\ &= \frac{2}{\pi}\left(\sum_{k=1}^{2n+1} \frac{1}{k}\right).\end{aligned}$$

Since the harmonic series $\sum_k \frac{1}{k}$ is divergent, it follows that the function is not of bounded variation.

Proposition 9. *If f, g are of bounded variation on $[a, b]$ so are $f + g$ and $f - g$.*

Proof. Observe that we have

$$\begin{aligned}\sum_{j=1}^{n} |f(x_j) \pm g(x_j) - [f(x_{j-1}) \pm g(x_{j-1})]| &\leq \sum_j |f(x_j) - f(x_{j-1})| + \sum_j |g(x_j) - g(x_{j-1})| \\ &\leq V(f, [a, b]) + V(g, [a, b]).\end{aligned}$$

□

Definition 10. Total variation of $f\colon [a, b] \to \mathbb{R}$, a function of bounded variation is the function F defined as follows:

$$F(x) := V(f, [a, x]) \equiv \text{lub}\,\{V(f, [a, x], P) : \text{ where } P \text{ is a partition of } [a, x]\}.$$

Theorem 11. *Let $f\colon [a, b] \to \mathbb{R}$ be of bounded variation and F its total variation. Then*
(i) $|f(x) - f(y)| \leq |F(x) - F(y)|$ *for* $a \leq x < y \leq b$.
(ii) F *and* $F - f$ *are increasing on* $[a, b]$ *and*
(iii) $V_a^b(f) \leq V_a^b(F)$.

Proof. The idea is very simple. If $x < y$, then any partition $P = \{x_0, \ldots, x_n = x\}$ of $[a, x]$ gives rise to a partition Q of $[a, y]$ in a natural way: $Q = \{a = x_0, \ldots, x_n = x, y\}$. Hence we obtain

$$\begin{aligned}\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| &\leq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + |f(y) - f(x)| \\ &\leq F(y).\end{aligned}$$

If we take the LUB of this inequality over all the partitions of $[a, x]$ we obtain

$$F(x) \leq F(x) + |f(y) - f(x)| \leq F(y).$$

(i) follows from this.

If $\emptyset \neq A \subset B \subset \mathbb{R}$, then we know $\text{lub}\, A \leq \text{lub}\, B$. Given any partition P of $[a, x]$, let Q denote the partition of $[a, y]$ as in (i). Then we have $V(f, [a, x], P) \leq V(f, [a, y], Q)$. If we take the LUB of this inequality over all the partitions of $[a, x]$, we get $F(x) \leq \text{lub}\, \{V(f, [a, y], Q)\} \leq F(y)$. Hence it follows that F is increasing on $[a, b]$.

To show that $F - f$ is increasing, we use the inequality of (i):

$$f(y) - f(x) \leq |f(y) - f(x)| \leq F(y) - f(x).$$

Therefore, $F(x) - f(x) \leq F(y) - f(y)$. Thus (ii) is proved.

Let P be a partition of $[a, b]$. Using (i), we obtain

$$\sum_j |f(x_j) - f(x_{j-1})| \leq \sum_j |F(x_j) - F(x_{j-1})| \leq V(F, [a, b]).$$

If we now take the LUB of this inequality over all partitions of $[a, b]$, we obtain $V(f, [a, b]) \leq V(F, [a, b])$. $\square$

Theorem 12 (Jordan). *Let $f\colon [a, b] \to \mathbb{R}$. Then f is of bounded variation iff there exist increasing functions F and G such that $f = F - G$.*

Proof. Take $G = F - f$ where F is the total variation of f. $\square$

Theorem 13. *Let $f\colon [a, b] \to \mathbb{R}^n$ be C^1. Then we have*

$$L(f) := V(f, , [a, b]) = \int_a^b \| f'(t) \| \, dt.$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. By the fundamental theorem of calculus, we have

$$\| f(x_j) - f(x_{j-1}) \| = \left\| \int_{x_j - 1}^{x_j} f'(t)\, dt \right\| \leq \int_{x_j - 1}^{x_j} \| f'(t) \| \, dt.$$

Summing this inequality over j we obtain

$$V(f, [a, b], P) \equiv \sum_j \| f(x_j) - f(x_{j-1}) \| \leq \int_a^b \| f'(t) \| \, dt.$$

Taking the LUB over all partitions, we get $V(f, [a, b]) \leq \int_a^b \| f'(t) \| \, dt$.

To get the reverse inequality, let $\varepsilon > 0$ be given. Since f' is continuous on $[a, b]$, it is uniformly continuous on $[a, b]$. Hence there exists $\delta > 0$ such that for $s, t \in [a, b]$ with $|s - t| < \delta$, we have $\| f'(s) - f'(t) \| < \varepsilon$. We subdivide $[a, b]$ into N subintervals of equal length $(b - a)/N$ in such a way that $(b - a)/N < \delta$. Let $P = \{x_0, x_1, \ldots, x_N\}$ be the partition. Observe that for $t \in [x_{j-1}, x_j]$, we have $\| f'(t) - f(x_j) \| < \varepsilon$. Hence for such t, we have

$$\| f'(t) \| \leq \| f'(x_j) \| + \varepsilon. \tag{1}$$

We integrate this inequality on $[x_{j-1}, x_j]$ and obtain

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \| f'(t) \| \, dt &\leq \int_{x_{j-1}}^{x_j} \| f'(x_j) \| \, dt + \varepsilon(x_i - x_{j-1}) \\
&= \| f'(x_j) \| (x_j - x_{j-1}) + \varepsilon(x_i - x_{j-1}) \\
&= \left\| \int_{x_{j-1}}^{x_j} \left[f'(t) + f'(x_j) - f'(t) \right] \, dt \right\| + \varepsilon(x_i - x_{j-1}) \\
&= \left\| \int_{x_{j-1}}^{x_j} f'(t) \, dt + \int_{x_{j-1}}^{x_j} \left[f'(x_j) - f'(t) \right] \, dt \right\| + \varepsilon(x_i - x_{j-1}) \\
&\leq \left\| \int_{x_{j-1}}^{x_j} f'(t) \, dt \right\| + \left\| \int_{x_{j-1}}^{x_j} \left(f'(x_j) - f'(t) \right) \, dt \right\| + \varepsilon(x_i - x_{j-1}) \\
&\leq \left\| \int_{x_{j-1}}^{x_j} f'(t) \, dt \right\| + \int_{x_{j-1}}^{x_j} \left\| \left(f'(x_j) - f'(t) \right) \right\| \, dt + \varepsilon(x_i - x_{j-1}) \\
&\leq \| f'(x_j) - f(x_{j-1}) \| + 2\varepsilon(x_i - x_{j-1}).
\end{aligned}$$

Summing over j, we arrive at

$$\int_a^b \| f'(t) \| \, dt \leq V(f, P) + 2\varepsilon(b - a) \leq V(f, [a, b]) + 2\varepsilon(b - a).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\int_a^b \| f'(t) \| \, dt \leq V(f, [a, b])$ and hence we conclude that $\int_a^b \| f'(t) \| \, dt = V(f, [a, b]) = L(f)$. □

We now prove that $\gamma = (x_1, \ldots, x_n)$ is rectifiable if each x_i is of "rectifiable" which is same saying that each x_i is of bounded variation. This follows from the equivalence of the standard (Euclidean) norm with the L^1 norm on $\mathbb{R}^N$: $\| x \|_1 := \sum_1^N |x_i|$. Note that we have $\| x \|_1 = \sum_j |x_j| \leq \sum_j \| x \| \leq N \| x \|$ for $x \in \mathbb{R}^N$. (In fact, using Cauchy-Schwartz, we have $\| x \|_1 \leq \sqrt{N} \| x \|$.) On the other hand, since $\| x \|_1^2 \geq \| x \|^2$, we have $\| x \| \leq \| x \|_1$.

Now if $L(x_i) = L_i$, for any partition P of $[a, b]$, we have

$$\begin{aligned}
L(\gamma, P) := \sum_j \| x(t_j) - x(t_{j-1}) \| &\leq \sum_j \| x(t_j) - x(t_{j-1}) \|_1 \\
&= \sum_j \sum_{i=1}^N |x_i(t_j) - x(t_{j-1})| \\
&= \sum_{i=1}^N \sum_j |x_i(t_j) - x(t_{j-1})| \\
&= \sum_i L(x_i, P) \\
&\leq \sum_i L_i.
\end{aligned}$$

This establishes our claim that γ is rectifiable iff each component function x_i of γ is "rectifiable" or is of bounded variation.