Jordan Curve Theorem

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

The aim of this article is to prove that if Γ is a subset of $\mathbb{R}^2$ that is homeomorphic to S^1, then $\mathbb{R}^2 \setminus \Gamma$ has two connected components. The proof uses some basic results in homotopy theory and covering spaces. It has a strong analytic flavour.

Definition 1. A *simple closed curve* or a *Jordan curve* in a topological space X is a one-to-one continuous map γ of S^1 into X. If X is hausdorff then γ is a homeomorphism of S^1 onto its image. The image $\Gamma := \gamma(S^1)$ is often called the simple closed curve.

The result we are after is stated formally as follows:

Theorem 2 (Jordan Curve Theorem)**.** *If γ is a simple closed curve in $\mathbb{R}^2$ with image Γ then $\mathbb{R}^2 \setminus \Gamma$ has precisely two connected components.*

Definition 3. Assuming the theorem, we can define the outside and the inside of Γ. The exterior of any disk $B(0, R)$ containing Γ is contained in a single connected component of $\mathbb{R}^2 \setminus \Gamma$. This component is called the *outside* of Γ and the other is called the *inside* of Γ.

Thus our task is to prove that there is precisely one bounded component of $\mathbb{R}^2 \setminus \Gamma$, the inside of Γ.

We give two other formulations of the theorem.

Theorem 4. *If γ is a simple closed curve in S^2 with image Γ, then $S^2 \setminus \Gamma$ has precisely two connected components.*

Theorem 5. *Let $h\colon \mathbb{R} \to \mathbb{R}^2$ be a one-to-one continuous map such that $|h(t)| \to \infty$ as $|t| \to \infty$. Then $\mathbb{R}^2 \setminus h(\mathbb{R})$ has precisely two connected components.*

Thm. 5 implies Thm. 4: To see this, we regard S^2 as the one-point compactification of $\mathbb{R}^2$ and we regard S^1 as the one-point compactification of $\mathbb{R}$. Suppose γ is a simple closed curve in S^2, with image Γ. By means of a rotation of $\mathbb{R}^3$ we may assume that $\gamma(\infty) = \infty$. Then the restriction h of γ to $\mathbb{R}$ satisfies the hypothesis of Thm. 5. Since $S^2 \setminus \Gamma = \mathbb{R}^2 \setminus h(\mathbb{R})$, Thm. 5 shows that $S^2 \setminus \Gamma$ has precisely two connected components. Thus Thm. 4 is established.

Thm. 4 implies Thm. 2: This is very easy. For, we may regard a simple closed curve γ in $\mathbb{R}^2$ as a simple closed curve in $S^2 = \mathbb{R} \cup \{\infty\}$. If $S^2 \setminus \Gamma$ has two connected components U and V, and, say, $\infty \in V$, then $\mathbb{R}^2 \setminus \Gamma$ has two connected components, namely, U and $V \cup \{\infty\}$.

The following is the easiest version of van Kampen theorem.

Lemma 6. *Let U and V be simply connected, path connected open subsets of a path connected X. Assume further that $U \cap V$ is path connected. Then X is simply connected.*

Proof. Let $p \in U$ and $\gamma\colon [0,1] \to X$ be loop at p. Then $\gamma^{-1}(X \setminus U)$ is a compact subset of the open subset $\gamma^{-1}(V)$ of $[0,1]$. Using Lebesgue covering lemma we can show that there are finite number of disjoint closed intervals $I_j := [s_j, t_j]$, $1 \leq j \leq N$, of $[0,1]$ such that I_j's cover $\gamma^{-1}(X \setminus U)$, and $\gamma(I_j) \subset V$ for $1 \leq j \leq N$. In particular, $\gamma(s_j)$ and $\gamma(t_j)$ lie in $U \cap V$. By hypothesis, there is a path $\alpha_j\colon I_j \to U \cap V$ from $\gamma(s_j)$ to $\gamma(t_j)$. Let α be the path in U defined so that $\alpha(s) = \alpha_j(s)$ for $s \in I_j$, $1 \leq j \leq N$, and $\alpha(s) = \gamma(s)$ for $s \notin \cup I_j$. Since V is simply connected, the path α_j is homotopic to the restriction path of γ to I_j. Combining these homotopies in the obvious way, we see that α is homotopic to γ. Now the path α lies in U and U is simply connected, so that α is homotopic to a point. It follows that γ is homotopic to a constant path and hence X is simply connected. □

In fact, we need a stronger result (such as van Kampen theorem) than the last lemma. For instance, suppose X is path connected space such that $X = U \cup V$ with U and V simply connected. What can we say about $\pi_1(X)$? It turns out that $\pi_1(X)$ is a free group with number of generators one less than the number of components of $U \cap V$. However, luckily we need only the piece of information contained in Lemma 8.

The following exercise is meant to prepare the reader for the proof of the next result.

Ex. 7. We assume the figure eight is formed of two circles touching tangentially at a common point. The circles will be regarded as two loops based at the common point. Construct three sheeted covering of the figure eight to prove that the loops $\alpha \circ \beta$ and $\beta \circ \alpha$ lift to paths whose terminal points are different and hence the loops are not homotopic.

Lemma 8. *Let X be connected, locally path connected (hence path connected) space. Let U and V be simply connected subsets of X which cover X. If $U \cap V$ has three or more connected components, then $\pi_1(X)$ is not abelian.*

Proof. The proof depends on the construction of an appropriate covering space unwinding the two loops in X. (Have you done Exer. 7?)

Let W_1 and W_2 be distinct path components of $U \cap V$ and let $W_0 = (U \cap V) \setminus (W_1 \cup W_2)$. Then W_i, $0 \leq i \leq 2$ are disjoint open subsets of X whose union is $U \cap V$. By hypothesis, W_i's are nonempty.

Let U_{mn} and V_{mn} be disjoint copies of U and V respectively, for $m, n \in \mathbb{Z}$. An element of U_{mn} will be denoted by (x, U, m, n). Let Y denote the union of the U_{mn}'s and V_{mn}'s. Define an equivalence relation $\sim$ on Y so that $(x, U, j, k) \sim (y, V, m, n)$ iff one of the following condition holds:

(i) $x = y \in W_0$, $j = m$, $k = n$.
(ii) $x = y \in W_1$, $j = m + 1$, $k = n$.
(iii) $x = y \in W_2$, $j = m \neq 0$, $k = n$.
(iv) $x = y \in W_2$, $j = m = 0$, $k = n + 1$.

Thus the conditions (i)–(iv) give a prescription of pasting copies of U's with copies of V's. Let E denote the quotient space of Y with respect to this equivalence relation. The natural projections of U_{mn}'s and V_{mn}'s onto the first coordinates determine a natural projection π of E onto X. π is easily seen to be a covering map.

Fix $p \in U$. For $j = 1, 2$, let α_j be a loop in X that starts at p, goes through U to W_0 continues in V to W_j, and returns to p in U. One checks that the lift of α_1 to E starting at $(p, U, 0, 0)$ travels through U_{00}, then through V_{00}, then back through U_{10} to terminate at $(p, U, 1, 0)$. Similarly the lift of α_2 to E starting at $(p, U, 1, 0)$ terminates at $(P, U, 1, 0)$. Consequently the lift of $\alpha_1\alpha_2$ to E starting at $(p, U, 0, 0)$ terminates at $(p, U, 1, 0)$. On the other hand, the lift of $\alpha_2\alpha_1$ to E starting at $(p, U, 0, 0)$ terminates at $(p, U, 1, 1)$. Since the lifts of $\alpha_1\alpha_2$ and $\alpha_2\alpha_1$ terminate at different points of E, $\alpha_1\alpha_2$ is not homotopic to $\alpha_2\alpha_1$. Thus, $\pi_1(X)$ is not abelian. □

Lemma 9. *Let T be a proper closed subset of $\mathbb{R}^2$ and let Q be the open subset of $\mathbb{R}^3$ defined by $Q := \mathbb{R}^3 \setminus \{(x, y, 0) : (x, y) \in T\}$. Then*
(i) If $\mathbb{R}^2 \setminus T$ is connected then Q is simply connected.
(ii) If $\mathbb{R}^2 \setminus T$ has at least three connected components, then $\pi(Q)$ is not abelian.

Proof. Since $\mathbb{R}^2 \setminus T$ is not empty, Q is connected. Define subsets U and V by

$$\begin{aligned} U &= \{(x, y, z) : z > 0\} \cup \{(x, y, z) : z > -1, (x, y) \notin T\}, \\ V &= \{(x, y, z) : z < 0\} \cup \{(x, y, z) : z < 1, (x, y) \notin T\}. \end{aligned}$$

Then U and V are connected open subsets of Q that cover Q.

We claim that U and V are simply connected. It is enough to see that U is simply connected, since V is homeomorphic to U under the reflection $(x, y, z) \mapsto (x, y, -z)$.

Let $\gamma\colon [0, 1] \to U$ be a loop with $\gamma(0) = \gamma(1) = (0, 0, 1)$. Define $H\colon [0, 1] \times [0, 1] \to U$ by $H(t, s) := \gamma(t) + (0, 0, s - s \cdot z(t))$ where $z(t)$ is the z-coordinate of $\gamma(t)$ if this coordinate is negative and 0 if it is positive. It is easy to check that the image of H lies in U and that H is continuous. Also, $H(0, t) = \gamma(t)$ and $H(1, t) \subset \{(x, y, z) : z \geq 1\}$. Since $\{(x, y, z) : z \geq 1\}$ is a convex subset of U containing $(0, 0, 1)$, the loop $t \mapsto H(1, t)$ is homotopic in U to the constant loop at $(0, 0, 1)$. Therefore γ is homotopic in U to the constant loop at (0,0,1). Hence U is simply connected.

Next observe that

$$U \cap V = \{(x, y, z) : (x, y) \in T, -1 < z < 1\}$$

is homeomorphic to $(\mathbb{R}^2 \setminus T) \times (-1, 1)$. If $\mathbb{R}^2 \setminus T$ is connected, then $U \cap V$ is connected. Hence, by Lemma 6, Q is simply connected. On the other hand, if $\mathbb{R}^2 \setminus T$ has at least three components, the so does $U \cap V$. By Lemma 8, $\pi_1(Q)$ is not abelian. □

Lemma 10. *Let $h\colon \mathbb{R} \to \mathbb{R}^2$ be a one-to-one continuous map such that $|h(t)| \to \infty$ as $|t| \to \infty$. Let $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the natural embedding: $(x, y) \mapsto (x, y, 0)$. Then there is a homeomorphism F of $\mathbb{R}^3$ such that $(F \circ \iota \circ h)(t) = (0, 0, t)$.*

Proof. Define $g\colon h(\mathbb{R}) \to \mathbb{R}$ by $g(h(t)) = t$, $t \in \mathbb{R}$. It is easy to check that g is continuous on $h(\mathbb{R})$. By Tietze extension theorem, there is a continuous function $G\colon \mathbb{R}^2 \to \mathbb{R}$ such that $G(p) = g(p)$ for all $p \in h(\mathbb{R})$. Let h_i be the coordinate functions of h so that $h(t) = (h_1(t), h_2(t))$. Then G satisfies

$$G(h_1(t), h_2(t)) = t, \qquad t \in \mathbb{R}. \tag{1}$$

Define $F_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $F_1(x, y, z) := (x, y, z + G(x, y))$. Since F_1 has continuous inverse $(x, y, z) \mapsto (x, y, z - G(x, y))$, F_1 is a homeomorphism. Define $F_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $F_2(x, y, z) := (x - h_1(z), y - h_2(z), z)$. Since F_2 has continuous inverse $(x, y, z) \mapsto (x + h_1(z), y + h_2(z), z)$, F_2 is a homeomorphism.

Set $F := F_2 \circ F_1$, a homeomorphism of $\mathbb{R}^3$. Using Eq. 1, we obtain

$$(F \circ \iota \circ h)(t) = F_2(F_1(h_1(t), h_2(t), 0)) = F_2(h_1(t), h_2(t), 0) = t.$$

This proves the result. □

The following is an extension of the last lemma.

Ex. 11. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be closed subsets. Let $f \colon A \to B$ be a homeomorphism. Then there is a homeomorphism $\varphi \colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ such that, for $x \in A$, we have $\varphi(x, 0) = (0, f(x))$.

Proof. (of Thm. 5)
First note that $h(\mathbb{R}) \neq \mathbb{R}^2$. In fact, the condition on h ensures that h is a homeomorphism of $\mathbb{R}$ onto its image. If $h(\mathbb{R}) = \mathbb{R}^2$, then $\mathbb{R}$ and $\mathbb{R}^2$ are homeomorphic which is clearly false.

Consider the set Q of Lemma 9 for $T = h(\mathbb{R})$. The homeomorphism F of Lemma 10 maps Q homeomorphically onto the set $\mathbb{R}^3 \setminus \{(0, 0, t) : t \in \mathbb{R}\} \equiv (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}$. The fundamental group of this space is $\mathbb{Z}$. Hence $\pi_1(Q) \simeq \mathbb{Z}$. Since $\pi_1(Q)$ is nonzero and abelian, by Lemma 9 we see that $\mathbb{R}^2 \setminus h(\mathbb{R})$ has precisely two components. □

The above proof is an adaptation of Doyle's proof [1]. We give below his formulation and proof.

Theorem 12. *A simple closed curve J in $\mathbb{R}^2$ separates $\mathbb{R}^2$.*

Proof. If $\mathbb{R}^2 \setminus J$ is connected, compactify $\mathbb{R}^2$ by adding a point and then remove a point from J to get a topological closed line ℓ in $\mathbb{R}^2$ that does not separate $\mathbb{R}^2$.

Consider $\mathbb{R}^2$ as a plane in $\mathbb{R}^3$. By the van Kampen Theorem [3], $\mathbb{R}^3 \setminus \ell$ is simply connected. However by [2] there is a homeomorphism of $\mathbb{R}^3$ onto $\mathbb{R}^3$ carrying ℓ onto the z-axis. But $\mathbb{R}^3 \setminus z - \{\text{axis}\}$ has the homotopy type of a circle. Hence $\mathbb{R}^2 \setminus J$ is not connected. □

References

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[3] Munkres, *Topology*, Prentice Hall of India.