# Banach Mapping Theorem

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

**Theorem 1** (KNASTER FIXED POINT THEOREM)**.** *If $F\colon P(A) \to P(A)$ is order-preserving, meaning that $F(X)$ is contained in $F(Y)$ whenever $X \subset Y$, then $F$ has a fixed point.*

*Proof.* Let $A_1$ be the union of all sets $X \in P(A)$ such that $X \subseteq F(X)$. It is easy to see that $F(A_1) = A_1$. (In fact, this $A_1$ is the greatest fixed point; there is also a least one.) □

The following is a useful improvement of the usual Schroeder-Bernstein statement:

**Theorem 2** (BANACH MAPPING THEOREM)**.** *Given any mappings (not necessarily injections) $f\colon A \to B$ and $g\colon B \to A$, we can partition $A$ into disjoint sets $A_1$ and $A_2$, and $B$ into disjoint sets $B_1$ and $B_2$, so that $f[A_1] = B_1$ and $g[B_2] = A_2$.*

*Proof.* Define $F\colon P(A) \to P(B)$ by setting $F(X) = A \setminus g[B \setminus f[X]]$. Clearly, $F$ is order-preserving; by Knaster's theorem, then, $F$ has a fixed point. Let $A_1$ be a fixed point of $F$, and let $B_1 = f[A_1]$, $B_2 = B \setminus B_1$, and $A_2 = g[B_2] = A \setminus A_1$. □