A Lemma from Linear Algebra – Lagrange Mutipliers

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Theorem 1. *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} : \mathbb{R}^m \to \mathbb{R}^n$ *be a linear map induced by the* $n \times m$ *matrix* A *whose rows are* A_i, $1 \le i \le n$. *Let* $f: \mathbb{R}^m \to \mathbb{R}$ *be a linear map. Let* $\alpha \in \mathbb{R}^m$ *be the gradient of* f, *that is, the unique (row) vector* α *such that* $f(x) = \alpha \cdot x$ *for* $x \in \mathbb{R}^m$. *Then* $\ker A \subset \ker f$ *iff there exists scalars* $\lambda_i \in \mathbb{R}$, $1 \le i \le n$, *such that* $\alpha = \lambda_1 A_1 + \cdots + \lambda_n A_n$.

Proof. Let k be the rank of A. Asssume, without loss of generality, that $A_1, \ldots, A_k$ are (maximally) linearly independent. Define a $k \times m$ matrix A' whose rows are A_i, $1 \le i \le k$.

We claim: the kernels of the linear maps $A: \mathbb{R}^m \to \mathbb{R}^n$ and $A': \mathbb{R}^m \to \mathbb{R}^k$ are the same. Note that if $x \in \mathbb{R}^m$ is a (column) vector, then Ax is the column vector with i-th coordinate $A_i \cdot x$, $1 \le i \le n$. Similarly, $A'x$ is the column vector whose j-th coordinate is $A_j \cdot x$, $1 \le j \le k$. From this description, the claim is clear. For, if $A'x = 0$, then $A_j \cdot x = 0$ for $j = 1, \ldots k$. Hence if we write $A_r = c_{r1} A_1 + \cdots + c_{rk} A_k$, where $k + 1 \le r \le n$, then $A_r \cdot x = 0$. Hence $\ker A' \subset \ker A$. The other inclsuion is easy.

Suppose α is a not a linear combination of A_i, $1 \le i \le n$. Then it is not a lineat combination of A_j, $1 \le j \le n$. Hence the matrix B whose rows (in order) are $A_1, \ldots, A_k, \alpha$ has rank $k+1$ and hence as a linear map $B: \mathbb{R}^m \to \mathbb{R}^{k+1}$ is onto. In particular, there exists a vector $v \in \mathbb{R}^m$ such that $Bv = e_{k+1}$. This means that $A'v = 0$ and $\alpha \cdot v = 1$, that is, $v \in \ker A$ and $v \notin \ker f$, a contradiction. Hence α is a linear combination of A_i's. $\square$