Computation of Real Integrals using the Residue Theorem

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Abstract

This starts with a review of improper integrals and applies the residue theorem trick to compute five classes of real integrals. It also shows how the residue calculus is used in th etheory of Laplace transform and Fourier transforms, a topic most often neglected in standard books. Pictures are yet to be inserted.

Picture!

1 Preliminaries on Improper Integrals

Definition 1. Let $f: [a,\infty) \to \mathbb{R}$ be given. We say that $\int_a^\infty f$ exists if $\int_a^R f$ exists for all $R \geq a$ and if $\lim_{R\to\infty} \int_a^R f$ exists. If the latter limit is s, we then say $\int_a^\infty f$ converges to s and write $\int_a^\infty f = s$. We similarly assign a meaning to $\int_{-\infty}^a g$ for $g: (-\infty, a) \to \mathbb{R}$. Finally, we say that $\int_{-\infty}^\infty f$ exists and is s if $\int_{-\infty}^0 f = s_1$ and $\int_0^\infty f = s_2$ with $s = s_1 + s_2$.

The reader is advised to keep the analogous concept $\sum_{n=-\infty}^\infty a_n$ in the following.

Ex. 2. Let $f: \mathbb{R} \to \mathbb{R}$ be such that for any $\varepsilon > 0$, there exists $R > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x > R$ and $y > R$. Then $\lim_{x\to\infty} f(x)$ exists.

Proposition 3.
(1) *If f is continuous and if $\int_a^\infty |f|$ exists, then $\int_a^\infty f$ exists.*
(2) *Let f and g be continuous. Assume that $\int_a^\infty g$ exists and that there exists a positive constant such that $0 \leq f(x) \leq \lambda g(x)$ for $x \geq a$. Then $\int_a^\infty f$ exists.*
(3) *Let $f: \mathbb{R} \to \mathbb{R}$ be given. Then $\int_{-\infty}^\infty f = s$ iff for any $\varepsilon > 0$, there exists $R > 0$ such that $|\int_{-a}^b f - s| < \varepsilon$ for $a \geq R$ and $b \geq R$.*

Proof. To prove a part of (3), let $\varepsilon > 0$ be given. Let R be chosen as per the condition. Then for $a, b > R$, we have

$$|\int_a^b f| = |\int_{-R}^b - \int_{-R}^a f| < 2\varepsilon.$$

Thus, $\int_0^\infty f$ converges, say, to s_1. Similarly $\int_{-\infty}^0 f$ converges, say, to s_2. It remains to show that $s_1 + s_2 = s$. Given $\varepsilon > 0$, by the other part of (3), there exists $R > 0$ such that if $A > R$, then $|\int_{-A}^A f - (s_1 + s_2)| < \varepsilon$. Hence $|s - (s_1 + s_2)| < \varepsilon$. □

Ex. 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous such that there exist $R > 0$ and M such that $|f(x)| \leq \frac{M}{|x|^2}$ for all x with $|x| \geq R$. Then $\int_{-\infty}^{\infty} f$ exists.

Definition 5. There is another way of assigning a meaning to $\int_{-\infty}^{\infty} f$. We say that the *Cauchy principal value* of the integral is L if $\int_{-R}^{R} f$ exists for all R and $\lim_{R\to\infty} \int_{-R}^{R} f = L$. In such a case, we write $P.V. \int_{-\infty}^{\infty} f = L$. The next few exercises clarify the relation between these two concepts.

Ex. 6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be any odd continuous function: $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Then P.V. $\int_{-\infty}^{\infty} f = 0$. In particular, $\int_{-\infty}^{\infty} x$ is not convergent but its Cauchy principal value exists.

Ex. 7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be even: $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Assume that the principal value P.V.$\int_{-\infty}^{\infty} f$ exists. Then $\int_{-\infty}^{\infty} f$ exists. *Hint:* $\int_{-R}^{0} f = \int_{0}^{R} f = \frac{1}{2}\int_{-R}^{R} f$.

Ex. 8. Let $\int_{-\infty}^{\infty} f$ exist. Then P.V. $\int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} f$.

Remark 9. Most often, we may first of all establish the convergence of $\int_{-\infty}^{\infty} f$ and then compute P.V. $\int_{-\infty}^{\infty} f$. In view of Ex. 8, we then would have computed $\int_{-\infty}^{\infty} f$.

There is another kind of improper integrals: these are the ones where the integrand becomes infinite at some points of the interval of integration. Let f be piecewise continuous on $[a, b]$, except, say, at $c \in [a, b]$. If the limits $\lim_{\varepsilon\to 0_+} \int_{a}^{c-\varepsilon} f(x)\, dx$ and $\lim_{\delta\to 0_+} \int_{c+\delta}^{b} f(x)\, dx$ exist, we then define

$$\int_a^b f(x)\, dx := \lim_{\varepsilon\to 0_+} \int_a^{c-\varepsilon} f(x)\, dx + \lim_{\delta\to 0_+} \int_{c+\delta}^{b} f(x)\, dx.$$

The integral is then called an improper integral.

Ex. 10. Define the analogous notion of principal value of such type of integrals.

Ex. 11. How do you define $\int_a^b f(x)\, dx$ if f becomes infinite at some point?

Example 12. The integral $\int_{-1}^{1} \frac{1}{x}$ does not exist but the principal value

$$\text{P.V.} \int_{-1}^{1} \frac{1}{x} \text{ exists and is } 0.$$

What is the principal value of $\int_{-1}^{2} \frac{1}{x}\, dx$?

2 Evaluation of Real Integrals – Type 1

Integrals of the form $\int_{-\infty}^{\infty} f$ where $|f(x)| \leq \frac{C}{|x|^2}$ for $|x|$ very large. We start with a simplest example.

Example 13. Let us consider $\int_{-\infty}^{\infty} \frac{1}{1+x^2}$. By Ex. 4, $\int_{\mathbb{R}} f$ exists. Let $f(z) := \frac{1}{1+z^2}$. The poles of f are at $z = \pm i$. Let C_R be the closed path: the line segment $[-R, R]$ followed by $\gamma_R(t) := Re^{it}$, $0 \leq t \leq \pi$.

If $R > 1$, then the pole $z = i$ is inside γ. Hence by the residue theorem we have

$$\int_{C_R} f = 2\pi i \mathrm{Res}\,(f; i) = 2\pi i \times \frac{1}{2i}.$$

Now, $\int_{C_R} f = \int_{-R}^{R} \frac{1}{1+x^2} + \int_{\gamma_R} f$. We claim that the second integral on the right goes to 0, as $R \to \infty$. The result then follows from Remark 9. Now, the claim follows from the estimate below:

$$\begin{aligned} \left|\int_{\gamma_R} f(z)\, dz\right| &= \left|\int_0^{\pi} \frac{iRe^{i\theta}}{1 + R^2 e^{2i\theta}}\, d\theta\right| \\ &\leq \int_0^{\pi} \frac{R}{R^2 - 1}\, d\theta. \end{aligned}$$

Example 14. Consider $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\, dx$,where $a > 0$. We let $f(z) := \frac{z^2}{(z^2+a^2)^2}$. Then f has poles at $z = \pm ia$. We find $\mathrm{Res}\,(f; ia) = \frac{1}{4ia}$. If C_R is as in Example 13 with $R > a$, we then have

$$\int_{C_R} f = 2\pi i \mathrm{Res}\,(f; ia) = 2\pi i \times \frac{1}{4ia}.$$

It is easy to show that $\int_{\gamma_R} f \to 0$ as $R \to \infty$. We conclude that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} = \frac{\pi}{2a}$. (How?)

These examples suggest the following result whose proof is left to the reader.

Theorem 15. *Let f be holomorphic on $\mathbb{C}$ except at a finite number of points, none of which are real and that those in the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ are $z_1, \ldots, z_n$. Assume that* *there exist positive constants R and M such that $|z^2 f(z)| \leq M$ for $|z| \geq R$. Then*

$$\int_{-\infty}^{\infty} f = 2\pi i \times \sum_{k=1}^{n} \operatorname{Res}(f; z_k).$$

□

Ex. 16. Prove the following.
(1) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \pi \cdot \frac{(2n)!}{2^{2n}(n!)^2}$.
(2) $\int_0^{\infty} \frac{1}{(1+x^2)^2(x^2+4)}\, dx = \frac{\pi}{6}$.
(3) $\int_0^{\infty} \frac{1}{1+x^n}\, dx = \frac{\pi/n}{\sin(\pi/n)}$. Deduce $\int_0^{\infty} \frac{dx}{1+x^{2n}} = \frac{\pi}{2n \sin \frac{\pi}{2n}}$.
(4) $\int_0^{\infty} \frac{2x^2-1}{x^4+5x^2+4} = \frac{\pi}{4}$.

Example 17. The integral $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}$ exists, for $a > 0$. The obvious choice $f(z) := \frac{\cos z}{z^2+a^2}$ is bad. We consider $f(z) := \frac{e^{iz}}{z^2+a^2}$. Let C_R be as earlier. Then the only pole inside C_R is at $z = ia$, since $a > 0$. The residue $\operatorname{Res}(f; ia) = \frac{e^{-a}}{2ia}$. Proceeding as usual, we see that the given real integral converges to $\frac{\pi e^{-a}}{a}$, provided that we show that $\int_{\gamma_R} f \to 0$ as $R \to \infty$. On γ_R, we note that $|e^{iz}| = e^{-y} \leq 1$ so that

$$|\int_{\gamma_R} \frac{e^{iz}}{z^2+a^2}| \leq \pi R \frac{1}{R^2-a^2}, \qquad \text{for } R > a$$

which is what we wanted. Note that had we chosen cos in place of e^{iz}, we would be in trouble, since $|\cos z|$ becomes large when $\operatorname{Im} z$ is large.

We deduce that $\int_{-\infty}^{\infty} \frac{\cos 2x}{1+x^2} = \frac{\pi}{e^2}$ and $\int_{-\infty}^{\infty} \frac{\sin 2x}{1+x^2} = 0$. (To prove the second, do you need residue theory?)

Example 18. $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx = \frac{\pi}{e}$.

3 Evaluation of Real Integrals – Type 2

We now consider the integrals of the form $\int_{-\infty}^{\infty} f(x) \cos ax$ etc. The following result says that the integral converges if $f(z) \to 0$ as $|z| \to \infty$ in the upper half plane. Note that we cannot invoke Ex. 4 here.

Theorem 19 (Jordan's Lemma)**.** *Let f be holomorphic on $\mathbb{C}$ except at a finite number of points, none of which are real and that those in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are $z_1, \dots, z_n$. Assume that given $\varepsilon > 0$ there exists $R > 0$ such that $|f(z)| < \varepsilon$ whenever $\operatorname{Im} z > R$. Let $a > 0$. Then*

$$\int_{-\infty}^{\infty} f(x)e^{iax}\, dx = 2\pi i \times \sum_{k=1}^{n} \operatorname{Res}(g; z_k),$$

where $g(z) = f(z)e^{iaz}$.

Proof. Given $\varepsilon > 0$, choose $R > 0$ such that (i) $|z_k| < R$ for $1 \le k \le n$, (ii) $|f(z)| \le \varepsilon$ for z in the upper half plane with $\operatorname{Im} z > R$ and (iii) $te^{-at} \le 1$ for $t \ge R$. Let $a > R$, $b > R$ and $c := a + b$. Choose C to be the rectangle with vertices at $-a$, b, $b + ic$ and $-a + ic$.

Then by residue theorem,

$$\int_C f(z)e^{iz} = 2\pi i \times \sum_{k=1}^{n} \operatorname{Res}(g, z_k).$$

Let the line segments $[b, b + ic]$, $[b + ic, -a + ic]$ and $[-a + ic, -a]$ be denoted by γ_1, γ_2 and γ_3 respectively. For z on γ_1 and γ_3, we have $|f(z)| \le \varepsilon$ and $|e^{iaz}| = e^{-ay}$ so that

$$|\int_{\gamma_j} g| \le \varepsilon \int_0^c e^{-ay}\, dy = \frac{\varepsilon}{a}(1 - e^{-ac}) \le \frac{\varepsilon}{a}, \qquad \text{for } j = 1, 3.$$

Since $L(\gamma_2) = c$, we have

$$|\int_{\gamma_2} g| \le c\varepsilon e^{-ac} \le \varepsilon.$$

Now the result follows from Prop. 3 (3). □

The following is a more popular version of Jordan's lemma which is given below. Most often when it is used what we get is the Cauchy principal value of $\int_{-\infty}^{\infty}$. So one needs some standard test of convergence of improper integral to assert the existence of the integral under consideration.

Lemma 20 (Jordan)**.** *Let $\gamma_R(t) := Re^{it}$, $0 \le t \le \pi$. Assume that (1) f is continuous on γ_R for $R \ge R_0$, (2) $|f(z)| \le M_R$ on γ_R, where $M_R \to 0$ as $R \to \infty$ and (3) $a > 0$. Then*

$$\int_{\gamma_R} f(z)e^{iaz} \to 0, \qquad \text{as } R \to \infty.$$

Proof. Proceeding as usual, we arrive at

$$|\int_{\gamma_R} f(z)e^{iaz}| \le R\int_0^{\pi} M_R e^{-aR\sin t}\,dt = 2R\int_0^{\pi/2} M_R e^{-aR\sin t}\,dt.$$

The crucial observation now is the following estimate (Ex. 21 below):

$$\sin t \ge \frac{2}{\pi}t, \qquad \text{for } 0 \le t \le \pi/2. \tag{1}$$

Using this, we get $|\int_{\gamma_R} f(z)e^{iaz}| \le \frac{\pi M_R}{a}(1 - e^{-aR})$. □

Ex. 21. Prove the Jordan's inequality:

$$\frac{2}{\pi} \le \frac{\sin t}{t} \le 1, \qquad \text{for } 0 \le t \le \pi/2.$$

Hint: To establish Jordan's inequality, observe that $\frac{\sin t}{t}$ decreases as its derivative $\frac{t\cos t - \sin t}{t^2} \le 0$. To see this, we show that the numerator is non-positive on $[0, \pi/2]$. To achieve this, again take derivative of the numerator to obtain $-t\sin t < 0$ on $[0, \pi/2]$. At $t = 0$, $t\cos t - \sin t = 0$ and hence the numerator is decreasing.

Example 22. We show that $\int_{-\infty}^{\infty} \frac{x\sin 3x}{x^2+9} = \frac{\pi}{2e^9}$. Consider $f(z) = \frac{ze^{3iz}}{z^2+9}$. Then f has a simple pole at $z = 3i$ in the upper half plane. Its residue there is $\operatorname{Res}(f; 3i) = \frac{\pi i}{2e^9}$. The result follows from the theorem on separating the real and imaginary parts.

Example 23. $\int_{\mathbb{R}} \frac{\cos x}{1+x+x^2}\,dx = \frac{2\pi}{\sqrt{3}}e^{-\frac{\sqrt{3}}{2}}\cos(1/2)$.

Example 24. $\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

Example 25. $\int_0^{\infty} \frac{e^{-2ix}}{1+x^4}\,dx = \pi 2\sqrt{2}(\cos\sqrt{2} - \sin\sqrt{2})e^{-\frac{1}{\sqrt{2}}}$.

Example 26. $\int_{\mathbb{R}} \frac{e^{ax}}{\cosh x}\,dx = \pi\sec(\pi a/2)$, where $-1 < a < 1$.

Ex. 27. Show the following:
(1) $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2} = \frac{\pi}{e^a}$.
(2) $\int_{-\infty}^{\infty} \frac{x^3\sin x}{(1+x^2)^2} = \frac{\pi}{2e}$.
(3) $\int_{-\infty}^{\infty} \left(\frac{x^2-a^2}{x^2+a^2}\right)\left(\frac{\sin x}{x}\right) = \pi(2e^{-a} - 1)$, where $a > 0$.
(4) $\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} = \pi(b - a)$, where $a, b \ge 0$.

4 Evaluation of Real Integrals – Type 3

These are the integrals of the form $\int_{-\infty}^{\infty} f$ where f has singularities on the real line.

Lemma 28 (Fractional Residue Theorem)**.** *Assume that f has a simple pole at a. Let $\gamma_r(t) := a + re^{it}$, $\alpha \le t \le \beta$. Then*

$$\int_{\gamma_r} f \to ia_{-1}(\beta - \alpha), \text{ as } r \to 0_+.$$

Remark 29. If the integration is performed on the full circle, then $\beta - \alpha = 2\pi$ and the residue theorem gives the value $2\pi i a_{-1}$, even without letting $r \to 0$. Thus the lemma indicates that the integration on an arc of the circle gives the corresponding fraction of $2\pi i a_{-1}$, provided that (i) $r \to 0$ and (ii) the pole is simple.

Proof. Without loss of generality, let us assume that $a = 0$.

Write $f(z) = a_{-1}z^{-1} + g(z)$ where g is holomorphic in a ball around a, say, $B[a, \varepsilon]$. Then $|g|$ is bounded on this compact set, say, by M. Thus for $0 < r < \varepsilon$, we have $|\int_{\gamma_r} g| \le M(\beta - \alpha)r$ which goes to 0 as $r \to 0$. It is easily seen that $\int_{\gamma_r} a_{-1}z^{-1}\, dz = ia_{-1}(\beta - \alpha)$. The result follows. ☐

Example 30. Consider $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$. The integrand is continuous and as such it has no singularity anywhere. However, if we consider $f(z) := \frac{e^{iz}}{z}$, then f has a simple pole at $z = 0$. We use the path as in the Jordan's lemma (Thm. 19) but with a slight detour near the pole $z = 0$. More precisely, we replace the bottom side of the square C by the path γ which is made up of the following: the line segment $[-a, -r]$, followed by the semicircular arc (in the lower half plane) $\gamma_r(t) = re^{it}$, $\pi \le t \le 2\pi$, the line segment $[r, b]$. Let the new closed path be denoted by σ_R.

Proceeding as in Jordan's lemma, we get:

$$\begin{aligned}2\pi = 2\pi i \times \operatorname{Res}(f;0) &= \int_{\sigma_R} f \\ &= \int_\gamma f + \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f \\ &\to \int_{-\infty}^{-r} f(x)\,dx + \int_r^\infty f(x)\,dx + \int_{\gamma_r} f(z)\,dz,\end{aligned}$$

as $R \to \infty$. By Lemma 28, $\int_{\gamma_r} f \to \pi i \operatorname{Res}(f;0)$, as $r \to 0$. Hence we find that $\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \pi$.

Remark 31. This example is very interesting on at least two counts:

(i) It is not at all clear at the outset why the integral exists. There is no trouble at all if the limits of integration are 0 and any real $R > 0$. The problem is at the far end, as $\int_R^\infty \frac{|\sin x|}{x}$ is divergent. See Ex. 32 below.

(ii) In view of Ex.32, it is clear that the function $\frac{\sin x}{x}$ is not Lebesgue integrable on $(0,\infty)$ but its improper (Riemann) integral exists!

Ex. 32. Show that $\int_0^\infty |\frac{\sin x}{x}|$ does not exist. *Hint:* Observe the following:

$$\begin{aligned}\int_0^\infty |\frac{\sin x}{x}|\,dx &> \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} |\frac{\sin x}{x}|\,dx \\ &\geq \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} |\frac{\sin x}{k\pi}|\,dx \\ &= \sum_{k=2}^n \frac{1}{k\pi} \int_0^\pi \sin x\,dx.\end{aligned}$$

Ex. 33. Use Lemma 20 to find P.V. $\int_{-\infty}^\infty \frac{\sin x}{x}\,dx$. *Hint:* Consider the path consisting of upper semicircle $\gamma_R(t) = Re^{it}$, $0 \le t \le \pi$, followed by the line segment $[-R,-r]$, the upper semicircle γ_r, the line segment $[r,R]$.

Show also that the integral $\int_{-\infty}^\infty \frac{\sin x}{x}\,dx$ is convergent and hence conclude that the principal value is the value of the integral.

Ex. 34. Prove the following:

(1) $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \pi/2$.

(2) $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$. *Hint:* $2\sin^2 x = 1 - \cos 2x$ and $f(z) = \frac{1-e^{2iz}}{z^2}$.

(3) $\int_0^\infty \frac{x-\sin x}{x^3}\,dx = \frac{\pi}{4}$. *Hint:* Consider $f(z) = \frac{z+ie^{iz}-i}{z^3}$.

(4) $\int_{-\infty}^\infty \frac{\cos x}{a^2-x^2}\,dx = \pi\frac{\sin a}{a}$, $a > 0$.

5 Evaluation of Real Integrals – Type 4

Trigonometric integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$.

Example 35. Let us compute $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ for $a > 1$ using the residue theorem. Let $z = e^{it}$. Then $\cos t = (z + \overline{z})/2 = \frac{z^2+1}{2z}$ and $dz = ie^{it}\, dt$. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. We have

$$\int_0^{2\pi} \frac{dt}{a+\cos t} = \int_\gamma \frac{2z}{z^2+2az+1} \frac{dz}{iz} = -2i \int_\gamma \frac{dz}{z^2+2az+1}.$$

The integrand f in the right most integral has simple poles at $-a \pm \sqrt{a^2-1}$. Of these, $-a-\sqrt{a^2-1}$ lies outside γ. Since the product of these poles is 1, the other pole $-a+\sqrt{a^2-1}$ lies inside γ. We find $\operatorname{Res}(f; -a+\sqrt{a^2-1}) = \frac{-i}{\sqrt{a^2-1}}$. Hence by the residue theorem, we have

$$\int_\gamma f = 2\pi i \times \operatorname{Res}(f; -a+\sqrt{a^2-1}) = \frac{2\pi}{\sqrt{a^2-1}}.$$

Example 36. Evaluate $\int_0^\pi \sin^{2n} t\, dt = \frac{1}{2}\int_0^{2\pi} \sin^{2n} t\, dt$. Since $\sin t = (e^{it} - e^{-it})/2i = (z - z^{-1})/2i$ for $z \in \gamma_1$, the unit circle, we have $\int_0^{2\pi} \sin^{2n} t\, dt = -i\int_{\gamma_1} f(z)$ where $f(z) = \frac{1}{z}(\frac{z-z^{-1}}{2i})^{2n}$. By binomial theorem,

$$\begin{aligned} f(z) &= z^{-1}\sum_{k=0}^{2n} \binom{2n}{k} (2i)^{-2n} z^k (-z)^{k-2n} \\ &= \sum_{k=0}^{2n} (-1)^{n-k} 4^{-n} \binom{2n}{k} z^{2k-2n-1}. \end{aligned}$$

Thus the only singularity of f in $B(0,1)$ is a pole at the origin. From the above expression, we have $\operatorname{Res}(f; 0) = 4^{-n}\binom{2n}{n}$. We find that $\int_0^\pi \sin^{2n} t\, dt = \pi 4^{-n}\binom{2n}{n}$.

It is possible to avoid the residue theorem in the above argument. Do you see how?

In general, given an integral of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)$, we can transform it into a path integral over the unit circle, by using the facts $\sin\theta = \frac{1}{2i}[z - z^{-1}]$ etc. on the unit circle. In fact, we have

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_{\gamma_1} f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}.$$

Ex. 37. Show that

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} = \int_0^{2\pi} \frac{d\theta}{1+a\sin\theta},$$

for $-1 < a < 1$.

Ex. 38. Show that

(1) $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}$, for $a > 1$.

(2) $\int_0^\pi \frac{a}{a^2+\sin^2\theta}\, d\theta = \frac{\pi}{\sqrt{1+a^2}}$, $a > 0$. *Hint:* Express $\sin^2 t$ in terms of $\cos 2t$.

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As in integral calculus, there are just too many tricks (some bordering on ingenuity) in the employment of residue theorem to compute the real integrals. We give a few examples of this kind.

Example 39. We show that $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n}\frac{1}{\sin(\pi/n)}$, for integers $0 < m < n$. Note that the integrals exist. We consider $f(z) = \frac{z^{m-1}}{1+z^n}$. The path under consideration is the sector formed of the line segment $[0, R]$, $R > 1$, followed by the arc $\gamma_R(t) = Re^{it}$, $0 \le t \le 2\pi/n$ and the line segment from $Re^{\frac{2\pi i}{n}}$ to 0.

The point $p = e^{i\pi/n}$ is only one pole inside this closed path. We have $\operatorname{Res}(f; p) = -\frac{1}{n}p^m$. It is easy to see that the integral over the arc goes to zero as $R \to \infty$. We compute the integral over the line segment $[Rp^2, 0]$ as follows: Let $\gamma_3(t) = tp^2$, $0 \le t \le R$. Then $\int_{[Rp^2,0]} f = -\int_{\gamma_3} f$.

$$-\int_{\gamma_3} f = -\int_0^R \frac{t^{m-1}p^{2m-2}}{1+t^n p^{2n}} p^2\,dt = -p^{2m}\int_0^R f(x),$$

that is, a multiple of the original integral. One can now proceed as usual and establish the claim.

Ex. 40. Derive the following special cases:

(1) $\int_0^\infty \frac{x^{n-1}}{1+x^{2n}} = \frac{\pi}{2n}$.

(2) $\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n}\frac{1}{\sin(\pi/n)}$.

Ex. 41. For $0 < a < 1$, show that $\int_{-\infty}^\infty \frac{e^{ax}}{e^x+1} = \frac{\pi}{\sin \pi a}$ by integrating along the rectangle whose vertices are $-R$, R, $R + 2\pi i$ and $-R + 2\pi i$. *Hint:* The integrals along the vertical sides go to zero as $R \to \infty$ while the one on the top side is a multiple of the given integral. (Did you show that the integral exists?)

Ex. 42. Evaluate $\int_0^\infty \frac{t^{a-1}}{1+t}\,dt$. *Hint:* Make the substitution $t = e^x$. Use the last exercise.

Ex. 43. In general, integrals of the form $\int_0^\infty x^{-a-1} f(x)\,dx$ can be evaluated as follows: Let $x = e^t$ and the integral becomes $\int_{-\infty}^\infty e^{at} f(e^t)\,dt$. Now let $g(z) := \exp(az)f(e^z)$, $z \in \mathbb{C}$. Integrate g round the boundary of the rectangle with vertices at R, $R + 2\pi i$, $-T + 2\pi$ and $-T$ and take limits as $R, T \to \infty$. Use this method to show:

(i) $\int_0^\infty \frac{x^{a-1}}{1+x^b}\,dx = \frac{\pi}{b \sin(\frac{\pi a}{b})}$, $0 < a < b$.

(ii) $\int_0^\infty \frac{x^a}{1+2x\cos\theta+x^2}\,dx = \frac{\pi}{\sin(\pi a)}\frac{\sin a\theta}{\sin\theta}$, $-1 < a < 1$ and $-\pi < \theta < \pi$.

6 Fresnel's and the Probability Integrals

Example 44 (Fresnel's and the probability integrals)**.** We compute the so-called Fresnel's integrals $\int_0^\infty \cos(x^2) = \int_0^\infty \sin(x^2)\,dx$ in two different ways and show that each of them is $\frac{\sqrt{\pi}}{2\sqrt{2}}$. The first method relates them to the probability integral. As a by-product, we also compute the probability integral. This approach is due to Cadwell.

Let us integrate the entire function $f(z) := e^{iz^2}$ along γ where γ is the juxtaposition of three paths: $[0, R]$, $A_R := Re^{it}$, $(0 \le t \le \pi/4)$ and $[Re^{i\pi/4}, 0]$. Draw pictures.

By Cauchy's theorem, we have $\int_\gamma f = 0$ so that

$$\int_0^R e^{it^2}\,dt + \int_{A_R} e^{iz^2}\,dz - \int_0^{Re^{i\pi/4}} e^{iz^2}\,dz = 0.$$

On the arc A_R, we have $z = Re^{it}$ so that

$$\begin{aligned}
|\int_{A_R} e^{iz^2}\,dz| &= |\int_0^{\pi/4} e^{iR^2(\cos 2t + i\sin 2t)} \cdot iRe^{it}\,dt| \\
&\le R\int_0^{\pi/4} e^{-R^2\sin 2t}\,dt \\
&= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\theta}\,d\theta \\
&\le \frac{R}{2}\int_0^{\pi/2} e^{-\frac{2R^2}{\pi}} \\
&= \frac{R}{2}\frac{\pi}{2R^2}(1 - e^{-R^2}).
\end{aligned}$$

On $[Re^{i\pi/4}, 0]$, we have $z = re^{i\pi/4}$, $0 \le r < R$ so that $e^{iz^2} = e^{-r^2}$. Hence, as $R \to \infty$, we obtain

$$\int_0^\infty e^{ix^2}\,dx = \int_0^\infty e^{-r^2} e^{i\pi/4}\,dr = e^{i\pi/4}\int_0^\infty e^{-r^2}\,dr.$$

On equating the real and imaginary parts, we obtain

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}\,dx. \tag{2}$$

Let us now employ a different path and a function with a simple pole to compute the first two integrals of (6). Let $A = -\sqrt{\pi}/2 - iR$, $B = \sqrt{\pi}/2 - iR$, $C = \sqrt{\pi}/2 + iR$ and $D = -\sqrt{\pi}/2 + iR$. Let γ be the rectangle consisting of $[A, B]$, $[B, C]$ $[C, D]$ and $[D, A]$. We integrate $f(z) = \frac{e^{iz^2}}{\sin\sqrt{\pi}z}$ along this path. A trivial computation shows that

$$\int_{[B,C]} f(z)\,dz + \int_{[D,A]} f(z)\,dz = 2\int_{-R}^{R} e^{i(\frac{\pi}{4}-y^2)} i\,dy.$$

On $[C, D]$ we have $|\sin z\sqrt{\pi}| > \sinh\frac{R}{\sqrt{\pi}}$ and $|e^{iz^2}| = e^{-2Rx}$. It follows that

$$\begin{aligned}
|\int_{[C,D]} f(z)\,dz| &\le \int_{-\sqrt{\pi}/2}^{\sqrt{\pi}/2} \frac{e^{-2Rx}}{\sinh R\sqrt{\pi}}\,dx \\
&= \frac{1}{\sinh R\sqrt{\pi}}\left[\frac{e^{-2Rx}}{-2R}\right]_{-\sqrt{\pi}/2}^{\sqrt{\pi}/2} \\
&= \frac{1}{R}.
\end{aligned}$$

Thus the path integral over $[C, D]$ goes to zero as $R \to \infty$. Similarly the path integral over $[A, B]$ goes to zero as $R \to \infty$. (Check it.)

We find that $\operatorname{Res}(f; 0) = \frac{1}{\sqrt{\pi}}$. Hence, by letting $R \to \infty$, we arrive at

$$\int_0^{\infty} e^{i(\frac{\pi}{4}-y^2)}\,dy = \frac{\sqrt{\pi}}{2}.$$

Equating the real and imaginary parts, we obtain

$$\int_0^{\infty} \sin(\frac{\pi}{4} - y^2)\,dy = 0 \text{ and } \int_0^{\infty} \cos(\frac{\pi}{4} - y^2)\,dy = \frac{\sqrt{\pi}}{2}.$$

Using the addition formulas for the sine and cosine functions we see that

$$\int_0^{\infty} \cos x^2\,dx = \int_0^{\infty} \sin x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{3}$$

In view of (6)–(7), we also deduce $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

7 Laplace Transform

Definition 45. Let $f\colon (0,\infty) \to \mathbb{C}$ be given. Let $s \in \mathbb{C}$. Then the Laplace transform of f is defined as follows:

$$\mathcal{L}(f(t)) \equiv F(s) := \int_0^\infty e^{-st} f(t)\, dt := \lim_{R\to\infty} \int_0^R e^{-st} f(t)\, dt,$$

if the limit exists.

Definition 46. A function $f\colon (0,\infty) \to \mathbb{C}$ is said to be of *exponential order* α if there exist $M > 0$, $\alpha \in \mathbb{R}$ and $T > 0$ such that

$$|f(t)| \leq Me^{\alpha t}, \text{ for } t \geq T.$$

We have the following easy theorem, whose proof is left as an exercise.

Theorem 47. *Let $f\colon [0,\infty) \to \mathbb{C}$ be piecewise continuous, of exponential order α. Then $F(s) = \mathcal{L}(f(t))$ exists for $s \in \mathbb{C}$ with $\operatorname{Re} s > \alpha$. Also, F is holomorphic in the region $\operatorname{Re} s > \alpha$.* □

We state (without proof) the following inversion theorem for Laplace transform of a function.

Theorem 48. *Let f be as in the last theorem. Let $\sigma > \alpha$. Then we can recover ffrom its Laplace transform F by the following 'inversion formula':.*

$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\sigma - iR}^{\sigma + iR} e^{ts} F(s)\, ds.$$

Residue theorem gives a very efficient method of computing the integral that appears in the inversion formula. The following theorem gives a useful method of computing the integral in the inversion formula.

Theorem 49. *Let F be holomorphic except for a finite number of poles, say, $\{z_j : 1 \leq j \leq n\}$. Assume that there exist $M > 0$ and $k > 0$ such that*

$$|F(s)| \leq M|s|^{-k}, \text{ for } |s| \gg 0.$$

Let $\alpha > 0$ be chosen such that all the poles lie on the left half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < \alpha\}$. Then for any $s > 0$, we have

$$\frac{1}{2\pi i} \lim_{R\to\infty} \int_{\alpha - iR}^{\alpha + iR} F(w) e^{sw}\, ds = \sum_{j=1}^{n} \operatorname{Res}\,(F(w)e^{sw}; z_j).$$

Proof. Let γ be the closed path composed of the line segment $[\alpha - iR, \alpha + iR]$ followed by the semicircle $C_R(t) := \alpha + Re^{it}$, $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. We choose $R \gg 0$ so that all the poles lie inside γ. We can apply the residue theorem to conclude

$$\int_\gamma F(w)e^{sw}\, dw = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\,(F(w)e^{sw}; z_j).$$

We claim that the integral $\int_{C_R} F(w)e^{sw}\, dw \to 0$ as $R \to \infty$. If we grant his, the result follows. The claim is proved in the next lemma. □

Lemma 50. *Let* $C_R(t) := \alpha + Re^{it}$, $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. *Let* $s > 0$. *Assume that* $|F(w)| \leq M|w|^{-k}$ *for* $|w| \gg 0$ *and* $k > 0$. *Then* $\int_{C_R} e^{ws}F(w) \to 0$ *as* $R \to \infty$.

Proof. We have

$$J = \int_{C_R} e^{ws}F(w)\,dw = \int_{\pi/2}^{3\pi/2} e^{\alpha s + Rse^{it}} F(w)Rie^{it}\,dt.$$

We have

$$|J| \leq \int_{\pi/2}^{3\pi/2} e^{\alpha s}e^{Rs\cos t}|F(w)Rie^{it}|\,dt.$$

Let us estimate a part of the integrand. Since $w = \alpha + Re^{it}$, we have $|w| \geq R - \alpha$. Hence,

$$|J| \leq Me^{\alpha s}(R-\alpha)^{-k}R\int_{\pi/2}^{3\pi/2} e^{Rs\cos t}\,dt.$$

The obvious trick here is to use Jordan's inequality.

$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1 \text{ for } 0 \leq \theta \leq \pi/2.$$

To make use of this, we make the change of variable $\theta = t + \frac{\pi}{2}$. The integral becomes $\int_0^\pi e^{-Rs\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-Rs\sin\theta}\,d\theta$. Hence, in view of Jordan's inequality,

$$\begin{aligned}
|J| &\leq 2Me^{\alpha s}(R-\alpha)^{1-k}\int_0^{\pi/2} e^{-Rs\frac{2}{\pi}\theta} \\
&= 2Me^{\alpha s}(R-\alpha)^{1-k}\left[1 - e^{-\frac{2Rs\theta}{\pi}}\right]_0^{\pi/2} \\
&= 2Me^{\alpha s}(R-\alpha)^{1-k}\frac{\pi}{2Rs}[1-e^{-Rs}] \to 0 \text{ as } R \to \infty.
\end{aligned}$$

□

8 Fourier Transforms

In this section, we define the Fourier transform of a function and compute the Fourier transform of two functions using complex analysis.

Definition 51. Given $f\colon \mathbb{R} \to \mathbb{C}$, its Fourier transform is defined by

$$(\mathcal{F}f)(s) \equiv \hat{f}(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-isx} f(x)\, dx, \ \ (s \in \mathbb{R}),$$

whenever the improper integral exists. Fourier transform exists on a large class of functions such as continuous functions vanishing at infinity or more generally all Lebesgue integrable functions.

Example 52. Let $f(x) = \frac{1}{1+x^2}$. Its Fourier transform is given by

$$\hat{f}(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-isx} \frac{1}{1+x^2}.$$

We compute this integral using the residue theorem. The natural choice to which the reside theorem is to be applied, is $f(z) = e^{-isz}(1+z^2)^{-1}$. The singularities are simples poles at $\pm i$. Now if we enclose one of these by a semicircle, $C_R(t) = Re^{it}$, where $0 \le t \le \pi$ or $\pi \le t \le 2\pi$, then the term $e^{-isw} = e^{-isR(\cos t + i \sin t)}$. Its absolute value is $e^{sR\sin t}$ which will go to zero as $R \to \infty$ if $s < 0$ and if $\sin t \ge 0$. This suggests that we consider the semicircle in the upper half-plane $\operatorname{Re} z \ge 0$.

If $s < 0$, then we would like $\sin t < 0$ so we shall consider the semi-circle in the lower half-plane in this case.

Assume $s < 0$ and $C_R(t) = Re^{it}$ for $0 \le t \le \pi$. Let γ be the path C_R followed by the line segment $[-R, R]$. We apply the residue theorem to the function $f(z) = e^{-isz}(1+z^2)^{-1}$ and the path γ. We obtain

$$\int_\gamma f(z) = 2\pi i \times \operatorname{Res}(f; i) = 2\pi i \times \frac{e^s}{2i} = \pi e^s.$$

An easy estimate shows that the integral over C_R goes to zero as $R \to \infty$. Hence $\int_\gamma f \to \int_{\mathbb{R}} f$ as $R \to \infty$. We thus get

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \times \pi e^s = \sqrt{\frac{\pi}{2}} e^s, \ \text{for } s < 0.$$

If $s > 0$, we employ the lower semicircle and obtain $\hat{f}(s) = \sqrt{\frac{\pi}{2}} e^{-s}$. Since $\hat{f}(0) = \sqrt{\frac{\pi}{2}}$, we have established

$$\hat{f}(s) = \sqrt{\frac{\pi}{2}} e^{-|s|}.$$

Example 53. Consider $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. We compute the Fourier transform of f using only Cauchy's theorem.

We have

$$\hat{f}(s) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(x^2+2isx)/2}\,dx$$
$$= \frac{1}{2\pi}e^{-s^2/2}\int_{\mathbb{R}} e^{-(x+is)^2/2}\,dx.$$

We claim that the last integral on the right is independent of s. (See Ex. **??**. We shall prove this below.) Assuming the claim for the moment, we arrive at

$$\hat{f}(s) = \frac{1}{2\pi}e^{-s^2/2}\int_{\mathbb{R}} e^{-x^2/2}\,dx = \frac{1}{2\pi}e^{-s^2/2}\times\sqrt{2\pi} = f(s).$$

Thus the Fourier transform of f is f itself.

We now prove the claim. Let γ be the juxtaposition of the paths $[-R_1, R_2]$, $[R_2, R_2 + is]$, $[R_2 + is, -R_1 + is]$ and $[-R_1 + is, -R_1]$. See Figure **??**. Since $f(z) = e^{-z^2/2}$ is holomorphic, we obtain $\int_\gamma f(z) = 0$.

$$0 = \int_{[-R1,R_2]} f + \int_{[R_2,R_2+is]} f + \int_{[R_2+is,-R_1+is]} f + \int_{[-R_1+is,-R_1]} f.$$

We claim that the second integral goes to zero as $R_2 \to 0$. We estimate the second integral using ML-inequality.

$$|\int_{[R_2,R_2+is]} f| = |\int_0^s e^{-(R_2+is)^2/2} i\,dt|$$
$$\leq |s|e^{s^2/2}e^{-R^2/2}.$$

The last quantity goes to zero as $R_2 \to \infty$. A similar argument shows that the fourth integral goes to zero as $R_1 \to \infty$. Thus we obtain

$$0 = \lim_{R_1,R_2\to 0}\left(\int_{-R_1}^{R_2} e^{-x^2/2}\,dx - \int_{-R_1}^{R_2} e^{-(x+is)^2/2}\,dx\right)$$
$$= \int_{-\infty}^{\infty} e^{-x^2/2}\,dx - \int_{-\infty}^{\infty} e^{-(x+is)^2/2}\,dx.$$

The claim is proved.

Example 54. We now compute the integral $\int_{\mathbb{R}} e^{-x^2}\,dx$ using the residue theorem. Note that the obvious choice is e^{-z^2} and it is entire. We learnt the computation below from [**?**] where it is credited to Kneser.

Consider

$$f(z) := \frac{e^{-z^2}}{1+e^{-2az}}, \text{ where } a = \sqrt{\pi}\cdot\frac{1+i}{\sqrt{2}} = \sqrt{\pi}e^{i\pi/4}.$$

Observe that $a^2 = \pi i$. Hence

$$e^{-2a(z+a)} = e^{-2az}e^{-2a^2} = e^{-2az}e^{-2\pi i} = e^{-2az}.$$

Therefor, a is a period of the function e^{-2az}. Let us compute

$$\begin{aligned} f(z) - f(z+a) &= \frac{e^{-z^2}}{1+e^{-2az}} - \frac{e^{-(z+a)^2}}{1+e^{-2a(z+a)}} \\ &= \frac{e^{-z^2} - e^{-z^2} e^{-2za} e^{-a^2}}{1+e^{-2az}} \\ &= \frac{e^{-z^2}(1+e^{-2az})}{1+e^{-2az}} \\ &= e^{-z^2}. \end{aligned} \tag{4}$$

The function f has simple poles at $-\frac{a}{2} + na$ where $n \in \mathbb{Z}$. Let us enclose the pole $-a/2$ between the $y = 0$ and the line through a parallel to $y = 0$. See Figure **??**. Let us compute the residue of f at $-a/2$.

$$\text{Res}\left(f; -\frac{a}{2}\right) = \frac{e^{-a^2/4}}{h'(-a/2)}, \text{ where } h(z) = 1 + e^{-2az}.$$

We compute $h'(-a/2)$:

$$h'(z) = -2ae^{-2az} \text{ so that } h'(-a/2) = -2ae^{a^2} = -2ae^{i\pi} = 2a.$$

We have $e^{-a^2/4} = e^{-i\pi/4}$ so that the residue is given by

$$\text{Res}\left(f; -\frac{a}{2}\right) = \frac{e^{-i\pi/4}}{2a} = \frac{1}{2}\frac{e^{-i\pi/4}}{\sqrt{\pi}e^{i\pi/4}} = \frac{1}{2}\frac{1}{\sqrt{\pi}}e^{-i\pi/2} = -\frac{i}{2\sqrt{\pi}}.$$

Note that the sum of the path integrals over the horizontal paths add up to $\int_{-R_1}^{R_2} e^{-x^2}\,dx$ in view of (4). Also, the integrals over the vertical paths go to zero as $R_1 \to \infty$ and $R_2 \to \infty$. Hence we obtain

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\pi i \times \text{Res}\left(f; -\frac{a}{2}\right) = 2\pi i \times \frac{-i}{2\sqrt{\pi}} = \sqrt{\pi}.$$

9 Integrals involving Multifunctions

The kind of integrals which are of the from $\int_0^\infty \frac{R(x)}{x^\alpha}\,dx$, $0 < \alpha < 1$ and $\int_0^\infty R(x)\log x\,dx$ fall under this type. We assume that $R(x) = P(x)/Q(x)$ is a rational function such that $\deg Q \geq 1 + \deg P$. We further assume that R has no pole in $[0, \infty)$.

Example 55. Let $R(x) = P(x)/Q(x)$ be a rational function. Assume that $\deg Q \geq 2 + \deg P$. We further assume that Q does not vanish on $(0, \infty)$ and that Q has a zero of order at most 1 at $x = 0$. Let $0 < a < 1$. We wish to evaluate the integral $\int_0^\infty R(x)x^a\,dx$. Note that the improper integral exists.

Let $U := \mathbb{C} \setminus \{t \in \mathbb{R} : t \geq 0\}$. Then U is star-shaped and hence a primitive of $1/z$ exists in U. Since any two primitives differ by a constant in U, we assume that the primitive, called again log is so chosen that $\log -1 = 1/e$. Note that this entails, for $x > 0$

$$\begin{aligned} \log(x + iy) &\to \log x, && \text{as } y \to 0_+ \\ \log(x - iy) &\to \log x + 2\pi i, && \text{as } y \to 0_+. \end{aligned}$$

Having chosen the logarithm in U, we define $z^\alpha := \exp(\alpha \log z)$. Choose $R \gg 0$ so that all the poles of $f(z) := R(z)z^{-\alpha}$ lie in $B(0, R)$. Let $r > 0$ and $\varepsilon > 0$ be small. We consider the closed key-hole path γ shown in the figure.

We have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{a \in U} \operatorname{Res}(f; a). \tag{5}$$

We claim that the integrals along the circular arcs $\int_{C_R} f \to 0$ as $R \to \infty$ and $\int_{C_r} f \to 0$ as $r \to 0$. Note that

$$|z^\alpha| = |e^{\alpha \log z}| = e^{\alpha \mathrm{Re} \log z} = e^{\alpha \log |z|} = |z|^\alpha.$$

Let us now attend to $\int_{C_r}$. Let r be sufficiently small. Since Q has a zero of order less than or equal to one, there exists $M > 0$ such that

$$|R(z)| \leq \frac{M}{|z|} \text{ for all } z \in B'(0, r).$$

Hence we arrive at

$$|\int_{C_r} f(z)\,dz| \le 2\pi r \times r^\alpha \frac{M}{r} \to 0, \text{ as } r \to 0.$$

Let us now attend to $\int_{C_R}$. Our hypothesis on the degrees of P and Q ensures the existence of $M > 0$ (different from the earlier M, if required!) such that

$$|R(z)| \le \frac{M}{|z|^2} \text{ for } |z| \gg 0.$$

We therefore conclude that

$$|\int_{C_R} f(z)\,dz| \le 2\pi R R^\alpha \frac{M}{R^2} \to 0 \text{ as } R \to \infty.$$

For fixed r and R, we have

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz \to (1 - e^{2\pi i\alpha}) \int_r^R e^{\alpha \log x} R(x)\,dx \text{ as } \varepsilon \to 0.$$

Thus, if we first let $\varepsilon \to 0$ and then let $r \to 0$ and $R \to \infty$, we arrive at

$$\int_0^\infty x^\alpha R(x)\,dx = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_{a \in U} \operatorname{Res}(f; a).$$

Example 56. Let $I := \int_0^\infty \frac{dx}{x^\alpha(1+x)}$ where $0 < \alpha < 1$. Then the only singularity of $f(z) = \frac{e^{-\alpha \log z}}{1+z}$ is a simple pole at $z = -1$. We find

$$\operatorname{Res}(f, z = -1) = e^{-\alpha \log(-1)} = e^{-\alpha\pi i}.$$

We therefore find

$$\int_0^\infty \frac{dx}{x^\alpha(1+x)} = \frac{2\pi i \times e^{-\alpha\pi i}}{1 - e^{-2\alpha\pi i}} = \frac{2\pi i \times e^{-\alpha\pi i}}{e^{-\alpha\pi i}(e^{\alpha\pi i} - e^{-\alpha\pi i})} = \frac{\pi}{\sin \pi\alpha}.$$

Example 57. $\int_0^\infty \frac{\sqrt{x}}{1+x^3}\,dx = \frac{\pi}{3}$.

Example 58. Let $I := \int_0^\infty \frac{\log^2 x}{1+x^2}\,dx$.

Let $U := \mathbb{C} \setminus \{t \in \mathbb{R} : t \le 0\}$. We define $\log z = \log|z| + i\theta$ where $\theta \in (-\pi, \pi)$.

We let $f(z) = \frac{\log^2 z}{1+z^2}$ and γ be the juxtaposition of the semi-circle C_R, $[-R, -r]$, C_r and $[r, R]$. As we have seen earlier, to make the path live in U, we need to lift $[-R, -r]$ followed by the left part of C_r by an ε-distance from the x-axis and then let $\varepsilon \to 0$. In practice, we do not do this but assume that our working justified along the lines of the last example. The point $z = i$ is the simple pole in the area enclosed by γ. The residue is given by $\operatorname{Res}(f; i) = \frac{(i\pi/2)^2}{2i} = -\frac{\pi^2}{8}$. By the residue theorem, we have Picture!

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_R^r \frac{\log(re^{i\pi})^2}{1+(re^{1\pi})^2} e^{i\pi}\,dr + \int_0^R \frac{\log^2 x}{1+x^2}\,dx + \int_{C_R} f(z)\,dz + \int_{C_r} f(z)\,dz \\ &= 2\pi i \times (-\frac{\pi^2}{8}) = -\frac{\pi^3}{8}. \end{aligned} \tag{6}$$

Using the ML estimate, we easily see that the path integrals along C_r (respectively the one along C_R) goes to zero as $r \to 0$ (respectively, as $R \to \infty$). Therefore, as $r \to 0$ and $R \to \infty$, we obtain from (6)

$$2\int_0^\infty \frac{\log x^2}{1+x^2}\,dx + 2\pi i \int_0^\infty \frac{\log x^2}{1+x^2}\,dx - \pi^2 \int_0^\infty \frac{1}{1+x^2}\,dx = -\frac{\pi^3}{4}.$$

Since $\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$, we find that

$$2\int_0^\infty \frac{\log x^2}{1+x^2}\,dx + 2\pi i \int_0^\infty \frac{\log x^2}{1+x^2}\,dx = \pi^2\frac{\pi}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}.$$

Equating the real and imaginary parts yields

$$\int_0^\infty \frac{\log^2 x}{1+x^2} = \frac{\pi^3}{8} \text{ and } \int_0^\infty \frac{\log x}{1+x^2} = 0.$$

Example 59. $\int_0^\infty \frac{\log x}{a^2+x^2}\,dx = \frac{\pi \log a}{2a}$ where $a > 0$.

Let $U := \mathbb{C} \setminus \{t \in \mathbb{R} : t \leq 0\}$. We define $\log z = \log|z| + i\theta$ where $\theta \in (-\pi, \pi)$. Then the function $f(z) := \frac{\log z}{z^2+a^2}$ is holomorphic on U except simple pole at ia.

We consider the path as in the Figure below: $C_R(t) := Re^{it}$, $t \in (\pi, -\pi)$, followed by $[-R - r]$, the semicircle C_r and finally $[r, R]$.

We have from the residue theorem

$$\int_\gamma f(z)\,dz = 2\pi i \times \operatorname{Res}(f; ai) = \frac{\pi \log a}{a} + i\frac{\pi^2}{2a}. \tag{7}$$

Proceeding in by now standard way, we arrive at the result.

Example 60. Use the key-hole path and the residue theorem to evaluate $\int_\gamma \frac{\log z}{(z+a)(z+b)}\,dz$ where a and b are positive. Hence conclude that $\int_0^\infty \frac{\log x}{(x+a)(x+b)} = \frac{\log(b/a)}{b-a}$. Of course, this can be solved using the standard partial fraction trick of calculus. Picture!

Example 61. Show that $\int_0^\infty \frac{\log x}{x^2-1}\,dx = \frac{\pi^2}{4}$. The obvious choice of function is $\frac{\log z}{z^2-1}$. The points to worry are at $z = \pm 1$ and $z = 0$. Since $\log z$ and $z^2 - 1$ have a simple zero at $z = 1$, we conclude that $z = 1$ is a removable singularity. The path o consider therefore is C_R and two smaller semicircles to avoid $z = -1$ and $z = 0$ on the line segment $[-R, R]$. It is easy to see the integrals over C_R and C_r go to zero as $R \to \infty$ and $r \to 0$. To tackle the contribution on the semicircle centered at $z = -1$, use the fractional residue theorem. Picture!