Lebesgue Covering Lemma

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Definition 1. Given an open cover $\{U_i : i \in I\}$ of a metric space (X, d), we say that a positive number δ is a *Lebesgue number* of the cover, if for any subset $A \subset X$ whose diameter is less than δ, there exists $i \in I$ such that $A \subset U_i$.

Remark 2. If δ is a Lebesgue number of the cover and $0 < \delta' \leq \delta$, then δ' is also a Lebesgue number of the given open cover.

In general, an open cover may not have a Lebesgue number.

Ex. 3. Let $X = (0, 1)$ with the usual metric. Let $U_n := (1/n, 1)$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. Does there exist a Lebesgue number for this cover?

Theorem 4 (Lebesgue Covering Lemma)**.** *Let (X, d) be a compact metric space. Let $\{U_i : i \in I\}$ be an open cover of X. Then a Lebesgue number exists for this cover.*

We give three proofs of this result.

Proof 1. For $x \in X$, there is an $i(x)$ such that $x \in U_{i(x)}$ and an $r(x) > 0$ such that $B(x, 2r(x)) \subset U_{i(x)}$. (Why?) There exist finitely many x_k, $1 \leq k \leq n$ such that $X = \cup_k B(x_k, r_k)$ where $r_k := r(x_k)$. Let δ be any positive real such that $\delta < \min\{r_k\}$. Let A be any subset with $\operatorname{diam}(A) < \delta$. Let $a \in A$. Then $a \in B(x_k, r_k)$ for some k. Let $x \in A$ be arbitrary. Then $d(x, x_k) \leq d(x, a) + d(a, x_k) < \delta + r_k < 2r_k$. Thus, $A \subset B(x_k, 2r_k) \subset U_{i(x_k)}$. □

Proof 2. Suppose that the result is not true. Then, for any $\delta = 1/n$, there is a subset A_n with diameter less than $1/n$ and such that it is not a subset of U_i for any i. Choose any $x_n \in A_n$. Then the sequence (x_n) has a convergent subsequence (x_{n_k}) such that $x_{n_k} \to p$ in X. Let $p \in U_i$. Let $r > 0$ be such that $B(p, 2r) \subset U_i$. Choose k so large that $x_{n_k} \in B(p, r)$ and $1/n_k < r$. Now if $a \in A_{n_k}$ is any element, then, $d(a, p) \leq d(a, x_{n_k}) + d(x_{n_k}, p) < 2r$. That is, $A_{n_k} \subset B(p, 2r) \subset U_i$, contradicting our assumption on the A_n's. □

Proof 3. We may assume that the given cover is finite, say, $\{U_i\}_{1 \leq i \leq n}$. Let $f_i(x) := d(x, X \setminus U_i)$. Then f_i are continuous and $f_i(x) = 0$ iff $x \in X \setminus U_i$, i.e., iff $x \notin U_i$. Let $f := \max\{f_i\}$. Then f is continuous on X and $f(x) = 0$ iff $x \notin U_i$ for all i, which is not possible, as U_i's cover X. Thus, $f(x) > 0$ for all $x \in X$. Let $\delta := \inf\{f(x) : x \in X\}$. Then $\delta > 0$. (Why?) Let A be any subset with $\operatorname{diam}(A) < \delta$. Let $a \in A$ be arbitrary. Then $f(a) \geq \delta$ and hence $f_i(a) \geq \delta$ for some i. Hence $a \in U_i$. If $x \in A$ is any point, then $x \in U_i$. For, otherwise, $x \in X \setminus U_i$ so that $d(a, x) \geq d(a, X \setminus U_i) = f_i(a) \geq \delta$. Hence $\operatorname{diam}(A) \geq \delta$, a contradiction. Hence $A \subset U_i$. □

Ex. 5. Let $f\colon (X,d) \to (Y,d)$ be continuous. Assume that X is compact. Prove that f is uniformly continuous (i) using the theorem and (ii) imitating the first (second) proof of the theorem.

Theorem 7 gives an interesting converse.

Definition 6. We say that a metric space has the Lebesgue number property, if every open cover of X has a Lebesgue number.

Theorem 7. *The following are equivalent for a metric space X:*

1. *X has Lebesgue number property.*
2. *Every continuous map from X to another metric space is uniformly continuous.*
3. *Every real valued continuous function on X is uniformly continuous.*

Proof. (1) $\implies$ (2): Let $f\colon X \to Y$ be a continuous map from X to another metric space Y. Let $\varepsilon > 0$ be given. Since f is continuous at $x \in X$, there exists a $\delta_x > 0$ such that for any $x' \in B(x, \delta_x)$, we have $d(f(x'), (x)) < \varepsilon/2$. Now, the collection $\{B(x, 2^{-1}\delta_x) : x \in X\}$ is an open cover of X. Let δ be a Lebesgue number of the cover. Let $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta/2$. Since the diameter of $B(x_1, \delta/2) \leq 2\delta$, there exists $x \in X$ such that $B(x_1, \delta/2) \subset B(x, 2^{-1}\delta_x))$. Hence $d(x_1, x_2) < \delta_x$. It follows that

$$d(f(x_1), f(x_2)) \leq d(f(x_1), f(x)) + d(f(x), f(x_2)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus f is uniformly continuous.

(2) $\implies$ (3): Take $Y = \mathbb{R}$ in (2).

(3) $\implies$ (1): We shall prove this contradiction. So, we assume that there exists an open cover $\{U_\alpha : \alpha \in I\}$ of X which has no Lebesgue number. This means that given any $n \in \mathbb{N}$, we can find $x_n \in X$ such that $B(x_n, 1/n)$ is not contained in any of the U_α's. In other words, given $n \in \mathbb{N}$, there exists $x_n \in X$ such that $B(x_n, 1/n) \setminus U_\alpha \neq emptyset$ for each $\alpha \in I$. We claim that no $B(x_n, 1/n)$ is a singleton. For, otherwise, x_n must be in some U_α. Hence $B(x_n, 1/n) = \{x_n\} \subset U_\alpha$, contradicting our choice of x_n. So, let $y_n \in B(x_n, 1/n)$ with $y_n \neq x_n$.

We claim that neither of the two sequences (x_n) and (y_n) can have a convergent subsequence. Assume the contrary. For instance, let us assume that (x_{n_k}) is a convergent subsequence of (x_n) converging to some $x \in X$. If $x \in U_\alpha$ (which must happen for some $\alpha \in I$), then there exists $r > 0$ such that $B(x, r) \subset U_\alpha$. Since $x_{n_k} \to x$ as $k \to \infty$, it follows that for some $k_0 \in \mathbb{N}$, we have $x_{n_k} \in B(x, r)$ for all $k \geq k_0$. Since $n_k \to \infty$, we see that $1/n_k < r - d(x, x_{n_k})$ for all sufficiently large k. As a consequence, $B(x_{n_k}, 1/n_k) \subset B(x, r) \subset U_\alpha$, a contradiction to our choice of x_n's. If (y_{n_k}) is a convergent subsequence, converging to $y \in X$, clearly, $x_{n_k} \to y$, impossible by what was seen just now.

We now construct two closed subsets A and B out of these two sequences inductively. Let $n_1 = 1$. We assume that x_1 and y_1 are already in A and B. We select $n_2 > n_1$ such that $x_{n_2} \neq x_{n_1}$ and $y_{n_1} \neq y_{n_1}$. This is possible, since otherwise, for all $n > n_1$, $x_n = x_1$ etc. Hence (x_n) and (y_n) will have convergent subsequences, contradicting our claim in the last paragraph. Assume that we have found $n_1 < \cdots < n_k$ such that $x_{n_1}, \ldots, x_{n_k}$ and $x_{n_1}, \ldots, x_{n_k}$ are disjoint. We select $n_{k+1} > n_k$ so that $x_{n_{k+1}} \notin \{x_{n_1}, \ldots, x_{n_k}\}$, $y_{n_{k+1}} \notin \{y_{n_1}, \ldots, y_{n_k}\}$ and the sets $A_k := \{x_{n_j} : 1 \leq j \leq k+1\}$ and $B_k := \{y_{n_j} : 1 \leq j \leq k+1\}$ are disjoint. That this is possible is seen as earlier.

> For, if this is not possible, then for all $n > n_k$, x_n must lie in the finite set $A_k \cup B_k$. It follows that x_n must be one of the elements in this set for infinitely many values of n. But then (x_n) will have a convergent subsequence.

We claim $A := \{x_{n_k}\}$ and $B := \{y_{n_k}\}$ are disjoint and closed. They are disjoint by construction. The set A is closed, if x is the limit of a sequence in A, then (x_n) has a convergent subsequence. Hence no $x \in X \setminus A$ could be the limit of a sequence in A. Hence A is closed. Similarly, B is closed. Let us define $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. Then f is continuous on the closed set $A \cup B$. By Urysohn's lemma, there exists a continuous extension, call it g, to X. The function g cannot be uniformly continuous. For, let $\delta > 0$ work for $\varepsilon = 1$. The $d(x_{n_k}, y_{n_k}) < 1/n_k < \delta$ for all sufficiently large k but $|g(x_{n_k}) - g(y_{n_k})| = 1$. So, we conclude that g is continuous but not uniformly continuous. This contradicts our hypothesis on X. □

Remark 8. Let X be a complete metric space such that every real valued continuous function is uniformly continuous. Is X compact? No, not necessarily. Look at $\mathbb{R}$ with discrete topology.