A Lebesgue Measurable Set which is not Borel

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Theorem 1. *There exists a (Lebesgue) measurable set which is not Borel.*

Proof. Assume, on the contrary, that every Lebesgue measurable set is a Borel set. Consider the Cantor function $f\colon [0,1] \to K$, where K is the Cantor set. Recall that in terms of binary and ternary expansions it is defined as

$$f(.b_1 b_2 \ldots (base 2)) = f\left(\sum_k \frac{b_k}{2^k}\right) = \sum_k \frac{2b_k}{3^k}.$$

We use, as is customary, non-terminating binary expansion. Since f is the pointwise limit of finite sums of characteristic functions of intervals, f is measurable. Observe that f is one-one.

Let E be a non-measurable subset of $[0,1]$. Now the set $f(E)$ is a subset of the Cantor set, which is of measure zero. Hence (due to completeness of the measure), we conclude that $f(E)$ is measurable and is of measure zero. Since by hypothesis every measurable set is Borel, $f(E)$ is also Borel. In view of the definition of measurable functions (which uses only the Borel sigma-algebra), $f^{-1}(f(E))$ is measurable. Since f is one-one, we have $f^{-1}(f(E)) = E$ and hence E is measurable, a contradiction. □