How to draw the graphs of functions on $\mathbb{R}^2$?

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Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given. How do we visualize the graph of f in $\mathbb{R}^3$? One thinks of such a graph as a surface in $\mathbb{R}^3$. We intersect this surface with the plane $z = c$ as $c \in \mathbb{R}$ varies and find the trace on the plane. (This could possibly be empty.) Knowing these traces will help us build the surface. To carry out this strategy, what we do is the following. Given $c \in \mathbb{R}$, we look at the set

$$f^{-1}(c) := L_c := \{(x, y) \in \mathbb{R}^2 : f(x, y) = c\}.$$

This set is called the *level set* or the *level curve* of f at the level c. We may consider this set L_c as an "implicitly defined" curve in the xy-plane. In fact, if the function f is a quadratic polynomial in x and y, then L_c is a (possibly empty) conic. To build the graph of f, all we have to do is to raise these level curves on the xy-plane, that is on the plane $z = 0$ to the plane $z = c$. As c varies over $\mathbb{R}$, we get the graph of f.

We shall illustrate this technique by showing a series of four pictures in a couple of examples below.

Example 1. Graph of $f(x, y) = x^2 - y^2$.

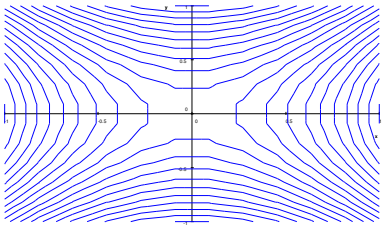

Figure 1: Level curves in the xy-plane

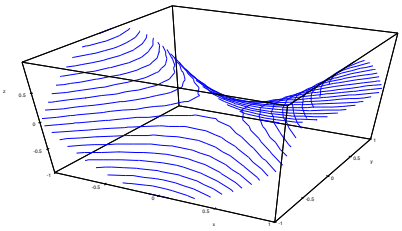

Figure 2: Level curves at various levels

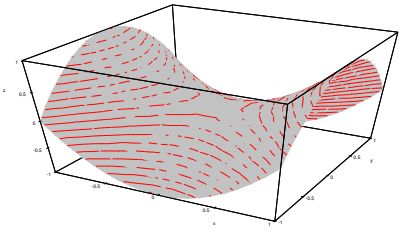

Figure 3: Level curves and the surface

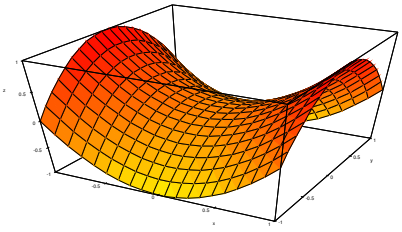

Figure 4: Graph of f

Example 2. Graph of $f(x, y) = e^{xy}$.

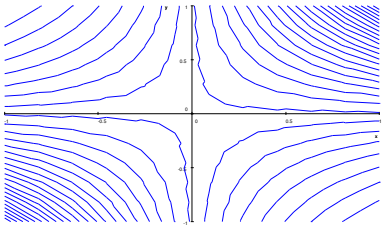

Figure 5: Level curves in the xy-plane

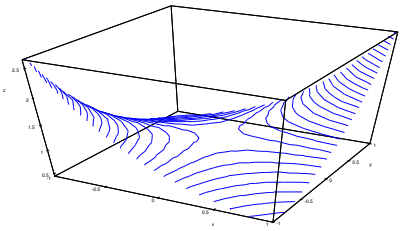

Figure 6: Level curves at various levels

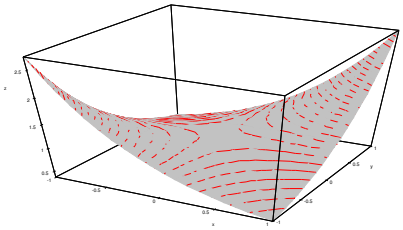

Figure 7: Level curves and the surface

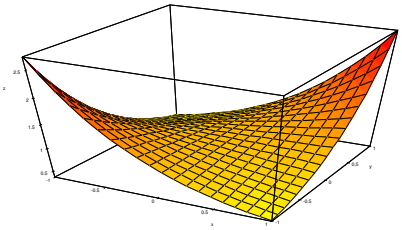

Figure 8: Graph of $f(x, y) = e^{xy}$

Example 3. Graph of $f(x, y) = x^2 + y^2$.

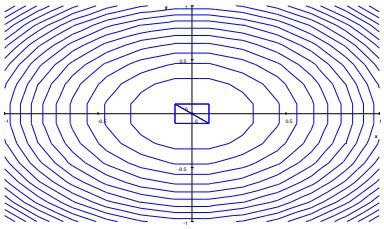

Figure 9: Level curves in the xy-plane

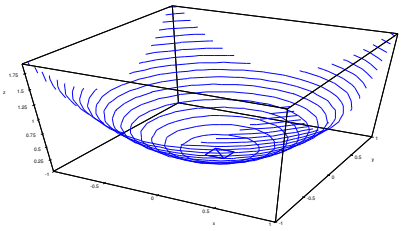

Figure 10: Level curves at various levels

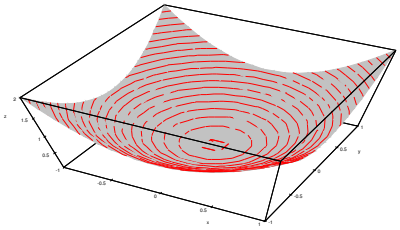

Figure 11: Level curves and the surface

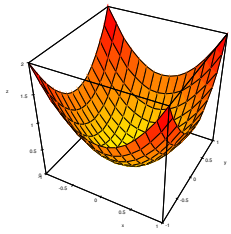

Figure 12: Graph of $f(x, y) = x^2 + y^2$